Discrete Mathematics 340 (2017) 963–968

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Note

Near packings of two graphs

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ARTICLE INFO

Article history:
Received 16 March 2016
Received in revised form 18 October 2016
Accepted 29 December 2016
Available online 8 February 2017

Keywords:
Packing
Near packing
Vertex stable graph

ABSTRACT

A *packing* of two graph G_1 and G_2 is a set $\{H_1, H_2\}$ such that $H_1 \cong G_1$, $H_2 \cong G_2$, and H_1 and H_2 are edge disjoint subgraphs of K_n. Let $\mathcal{F}$ be a family of graphs. A *$\mathcal{F}$-near-packing* of G_1 and G_2 is a generalization of a packing. In a $\mathcal{F}$-near-packing, H_1 and H_2 may overlap so the subgraph defined by the edges common to both of them is a member of $\mathcal{F}$. In the paper we study three families of graphs (1) $\mathcal{E}_k$ –the family of all graphs with at most k edges, (2) $\mathcal{D}_k$ –the family of all graphs with maximum degree at most k, and (3) $\mathcal{K}_k$ –the family of all graphs having the clique number at most k. By $m(n, \mathcal{F})$ we denote the smallest number m such that there are graphs G_1 and G_2 both of order n with $|E(G_1)| + |E(G_2)| = m$ which do not have a $\mathcal{F}$-near-packing. It is well known that $m(n, \mathcal{E}_0) = m(n, \mathcal{D}_0) = m(n, \mathcal{K}_1) = \lfloor \frac{3(n-1)}{2} \rfloor + 1$ because a $\mathcal{E}_0$, $\mathcal{D}_0$ or $\mathcal{K}_1$-near-packing is just a packing. In this paper we prove that $m(n, \mathcal{E}_1) = m(n, \mathcal{D}_1) = 2n - 1$ and, for sufficiently large n, $m(n, \mathcal{D}_2) = \lfloor \frac{5(n-1)}{2} \rfloor + 1$. We also obtain distinct bounds on $m(n, \mathcal{E}_k)$ for $k \geq 2$, $m(n, \mathcal{D}_3)$, and $m(n, \mathcal{K}_k)$ for $k \geq 2$, under conditions imposed on n and k.

1. Introduction

We use the term *graph* to refer to simple graphs without loops or multiple edges. The vertex set and edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. Two graphs G_1 and G_2 pack (into a complete graph K_n) if there is a pair of edge-disjoint subgraphs $\{H_1, H_2\}$ of K_n such that $H_1 \cong G_1$ and $H_2 \cong G_2$. Extremal problems on graph packing have been actively studied since the seventies. In 1978, Bollobás and Eldridge [2] and Sauer and Spencer [8] proved several important results. In particular Sauer and Spencer [8] showed the following theorem.

Theorem 1 ([8]). *Let G_1 and G_2 be two graphs on n vertices. If*

$$|E(G_1)| + |E(G_2)| \leq \left\lfloor \frac{3(n-1)}{2} \right\rfloor,$$

then G_1 and G_2 pack.

The bound in Theorem 1 is sharp.

In this paper we consider the following generalization of the notion of the packing. Let $\mathcal{F}$ be a family of graphs. We say that G_1 and G_2 have a *$\mathcal{F}$-near-packing* (into a complete graph K_n) if there is a pair of subgraphs $\{H_1, H_2\}$ of K_n with $H_1 \cong G_1$ and $H_2 \cong G_2$ and such that the subgraph defined by the common edges of H_1 and H_2 is a member of $\mathcal{F}$.

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http://dx.doi.org/10.1016/j.disc.2016.12.022

More technically, let G be a graph and W any set with $|V(G)| \leq |W|$. Given an injection $f : V(G) \to W$, let $f(G)$ denote the graph defined as follows

$$f(G) = (W, \{f(u)f(v) : uv \in E(G)\}) .$$

For two graphs G_1 and G_2 let

$$\begin{aligned} G_1 \oplus G_2 &= (V(G_1) \cup V(G_2), E(G_1) \cup E(G_2)) \\ G_1 \odot G_2 &= (V(G_1) \cap V(G_2), E(G_1) \cap E(G_2)) \end{aligned}$$

(note that $V(G_1)$ and $V(G_2)$ do not need to be disjoint).

Definition 1. Let $\mathcal{F}$ be any family of graphs and let G_1, G_2 be two graphs with $|V(G_1)| = |V(G_2)|$. A $\mathcal{F}$-near-packing of G_1 and G_2 is a bijection $f : V(G_1) \to V(G_2)$ such that $f(G_1) \odot G_2 \in \mathcal{F}$.

We consider three families of graphs : (1) $\mathcal{E}_k$ being the family of all graphs with at most k edges, (2) $\mathcal{D}_k$ being the family of all graphs with maximum degree at most k, and (3) $\mathcal{K}_k$ being the family of all graphs with the clique number at most k. By $m(n, \mathcal{F})$ we denote the smallest number m such that there exist two graphs G_1 and G_2, both of order n and satisfying $|E(G_1)| + |E(G_2)| = m$, which do not have a $\mathcal{F}$-near-packing. Since a $\mathcal{E}_0$, a $\mathcal{D}_0$ or a $\mathcal{K}_1$-near-packing is just a packing, Theorem 1 may be reformulated in the following way

Theorem 2 (*[8]*). $m(n, \mathcal{K}_1) = m(n, \mathcal{D}_0) = m(n, \mathcal{E}_0) = \lfloor 3(n-1)/2 \rfloor + 1$.

Near-packings were first mentioned (however, not under this name) by Bollobás and Erdős. They asked to determine $m(n, \mathcal{E}_k)$, see [1] pp. 436, Problem 24. Eaton [5] studied $\mathcal{D}_k$-near-packings, and the second author studied $\mathcal{F}$-near-packings of two copies of the same graph [9].

The notation is standard. For notions not defined here we refer to [3].

2. $\mathcal{E}_k$-near-packings

Theorem 3. $m(n, \mathcal{E}_k) \geq \sqrt{2(k+1)n(n-1)}$.

Proof. Let G_1, G_2 be two graphs of order n satisfying

$$|E(G_1)| + |E(G_2)| < \sqrt{2(k+1)n(n-1)}.$$

Thus,

$$|E(G_1)| \cdot |E(G_2)| < \left(\frac{\sqrt{2(k+1)n(n-1)}}{2} \right)^2 = (k+1)\binom{n}{2}. \tag{1}$$

Consider the probability space whose $n!$ points are all bijections $f : V(G_1) \to V(G_2)$. For any two edges $e_1 \in E(G_1)$, $e_2 \in E(G_2)$ let $X_{e_1e_2}$ denote the indicator random variable with value 1 if $f(e_1) = e_2$. Then

$$E(X_{e_1e_2}) = Pr[X_{e_1e_2} = 1] = \frac{2(n-2)!}{n!} = \binom{n}{2}^{-1}.$$

Let $X = \sum_{e_1 \in E(G_1), e_2 \in E(G_2)} X_{e_1e_2}$. Thus, by the linearity of expectation and by (1), we have

$$E(X) = \sum_{e_1 \in E(G_1), e_2 \in E(G_2)} E(X_{e_1e_2}) = |E(G_1) \parallel E(G_2)| \binom{n}{2}^{-1} < k+1.$$

This implies that there exists a bijection f such that $f(G_1) \odot G_2$ has at most k edges. Thus, f is a $\mathcal{E}_k$-near-packing of G_1 and G_2. $\square$

Corollary 4.

$$m(n, \mathcal{E}_1) = 2n - 1.$$

Proof. By Theorem 3, $m(n, \mathcal{E}_1) > 2(n-1)$. On the other hand the examples $G_1 = K_{1,n-1}$ and $G_2 = C_n$ show that $m(n, \mathcal{E}_1) \leq 2n - 1$. $\square$

Given two graphs G_1 and G_2 with $V(G_1) \cap V(G_2) = \emptyset$, the join $G_1 * G_2$ is a graph G with $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1), v \in V(G_2)\}$. Below we show that the bound in Theorem 3 is very near to be sharp.

Theorem 5. *Let $n_0(k)$ be sufficiently large. Let n be even with $n \geq n_0(k)$. Then*

$$m(n, \mathcal{E}_k) < \left(2\left\lceil \sqrt{k/2} \right\rceil + 1/2\right) n.$$