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Note

The numbers of edges of the order polytope and the chain polytope of a finite partially ordered set

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ABSTRACT

Let P be an arbitrary finite partially ordered set. It will be proved that the number of edges of the order polytope $\mathscr{O}(P)$ is equal to that of the chain polytope $\mathscr{C}(P)$. Furthermore, it will be shown that the degree sequence of the finite simple graph which is the 1-skeleton of $\mathscr{O}(P)$ is equal to that of $\mathscr{C}(P)$ if and only if $\mathscr{O}(P)$ and $\mathscr{C}(P)$ are unimodularly equivalent.

0. Introduction

In [5] the combinatorial structure of the order polytope $\mathscr{O}(P)$ and the chain polytope $\mathscr{C}(P)$ of a finite poset (partially ordered set) P is studied in detail. Furthermore, the problem when $\mathscr{O}(P)$ and $\mathscr{C}(P)$ are unimodularly equivalent is solved in [3]. In this paper it is proved that, for an arbitrary finite poset P, the number of edges of the order polytope $\mathscr{O}(P)$ is equal to that of the chain polytope $\mathscr{C}(P)$. Furthermore, it is shown that the degree sequence of the finite simple graph which is the 1-skeleton of $\mathscr{O}(P)$ is equal to that of $\mathscr{C}(P)$ if and only if $\mathscr{O}(P)$ and $\mathscr{C}(P)$ are unimodularly equivalent.

1. Edges of order polytopes and chain polytopes

Let $P = \{x_1, \ldots, x_d\}$ be a finite poset. Given a subset $W \subset P$, we introduce $\rho(W) \in \mathbb{R}^d$ by setting $\rho(W) = \sum_{i \in W} \mathbf{e}_i$, where $\mathbf{e}_1, \mathbf{e}_2 \ldots, \mathbf{e}_d$ are the canonical unit coordinate vectors of $\mathbb{R}^d$. In particular $\rho(\emptyset)$ is the origin of $\mathbb{R}^d$. A *poset ideal* of P is a subset I of P such that, for all x_i and x_j with $x_i \in I$ and $x_j \leq x_i$, one has $x_j \in I$. An *antichain* of P is a subset A of P such that x_i and x_j belonging to A with $i \neq j$ are incomparable. The empty set $\emptyset$ is a poset ideal as well as an antichain of P. We say that x_j *covers* x_i if $x_i < x_j$ and $x_i < x_k < x_j$ for no $x_k \in P$. A chain $x_{j_1} < x_{j_2} < \cdots < x_{j_\ell}$ of P is called *saturated* if x_{j_q} covers $x_{j_{q-1}}$ for $1 < q \leq \ell$.

The *order polytope* of P is the convex polytope $\mathscr{O}(P) \subset \mathbb{R}^d$ which consists of those $(a_1, \ldots, a_d) \in \mathbb{R}^d$ such that $0 \leq a_i \leq 1$ for every $1 \leq i \leq d$ together with

$$a_i \geq a_j$$

if $x_i \leq x_j$ in P.

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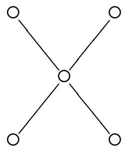

Fig. 1. The Poset X.

The *chain polytope* of P is the convex polytope $\mathscr{C}(P) \subset \mathbb{R}^d$ which consists of those $(a_1, \ldots, a_d) \in \mathbb{R}^d$ such that $a_i \geq 0$ for every $1 \leq i \leq d$ together with

$$a_{i_1} + a_{i_2} + \cdots + a_{i_k} \leq 1$$

for every maximal chain $x_{i_1} < x_{i_2} < \cdots < x_{i_k}$ of P.

One has $\dim \mathscr{O}(P) = \dim \mathscr{C}(P) = d$. The vertices of $\mathscr{O}(P)$ is those $\rho(I)$ for which I is a poset ideal of P ([5, Corollary 1.3]) and the vertices of $\mathscr{C}(P)$ is those $\rho(A)$ for which A is an antichain of P ([5, Theorem 2.2]). It then follows that the number of vertices of $\mathscr{O}(P)$ is equal to that of $\mathscr{C}(P)$. Furthermore, the volume of $\mathscr{O}(P)$ and that of $\mathscr{C}(P)$ are equal to $e(P)/d!$, where $e(P)$ is the number of linear extensions of P ([5, Corollary 4.2]).

In [4] a characterization of edges of $\mathscr{O}(P)$ and those of $\mathscr{C}(P)$ is obtained. Recall that a subposet Q of a finite poset P is said to be *connected* in P if, for each x and y belonging to Q, there exists a sequence $x = x_0, x_1, \ldots, x_s = y$ with each $x_i \in Q$ for which x_{i-1} and x_i are comparable in P for each $1 \leq i \leq s$.

Lemma 1.1. *Let P be a finite poset.*

(a) *Given poset ideals I and J with $I \neq J$, the segment combining $\rho(I)$ with $\rho(J)$ is an edge of $\mathscr{O}(P)$ if and only if $I \subset J$ and $J \setminus I$ is connected in P.*

(b) *Given antichains A and B with $A \neq B$, the segment combining $\rho(A)$ with $\rho(B)$ is an edge of $\mathscr{C}(P)$ if and only if $(A \setminus B) \cup (B \setminus A)$ is connected in P.*

Let, in general, G be a finite simple graph, i.e., a finite graph with no loop and with no multiple edge, on the vertex set $V(G) = \{v_1, \ldots, v_n\}$. The *degree* $\deg_G(v_i)$ of each $v_i \in V(G)$ is the number of edges e of G with $v_i \in e$. Let $i_1 \cdots i_n$ denote a permutation of $1, \ldots, n$ for which $\deg_G(v_{i_1}) \leq \cdots \leq \deg_G(v_{i_n})$. The *degree sequence* ([1, p. 216]) of G is the finite sequence $(\deg_G(v_{i_1}), \ldots, \deg_G(v_{i_n}))$.

Example 1.2. Let X denote the poset.

Then the degree sequence of the finite simple graph which is the 1-skeleton of $\mathscr{O}(X)$ is

$$(6, 6, 6, 6, 6, 6, 6, 6)$$

and that of $\mathscr{C}(X)$ is

$$(5, 6, 6, 6, 6, 6, 6, 7).$$

This observation guarantees that, even though the number of edges of $\mathscr{O}(X)$ is equal to that of $\mathscr{C}(X)$, one cannot construct a bijection $\varphi : V(\mathscr{O}(X)) \to V(\mathscr{C}(X))$, where $V(\mathscr{O}(X))$ is the set of vertices of $\mathscr{O}(X)$ and $V(\mathscr{C}(X))$ is that of $\mathscr{C}(X)$, with the property that, for α and β belonging to $V(\mathscr{O}(X))$, the segment combining α and β is an edge of $\mathscr{O}(X)$ if and only if the segment combining $\varphi(\alpha)$ and $\varphi(\beta)$ is an edge of $\mathscr{C}(X)$.

2. The number of edges of order polytopes and chain polytopes

We now come to the main result of the present paper.

Theorem 2.1. *Let P be an arbitrary finite poset. Then the number of edges of the order polytope $\mathscr{O}(P)$ is equal to that of the chain polytope $\mathscr{C}(P)$.*

Proof. Let Ω denote the set of pairs (I, J), where I and J are poset ideals of P with $I \neq J$ for which $I \subset J$ and $J \setminus I$ is connected in P. Let Ψ denote the set of pairs (A, B), where A and B are antichains of P with $A \neq B$ for which $(A \setminus B) \cup (B \setminus A)$ is connected in P.

As is stated in the proof of [4, Lemma 2.3], if there exist $x, x' \in A$ and $y, y' \in B$ with $x < y$ and $y' < x'$, then $(A \setminus B) \cup (B \setminus A)$ cannot be connected. In fact, if $(A \setminus B) \cup (B \setminus A)$ is connected, then there exists a sequence $x = x_0, y_0, x_1, y_1, \ldots, y_s, x_s = x'$ with each $x_i \in A \setminus B$ and each $b_j \in B \setminus A$ such that x_i and y_i are comparable for each i and that y_j and x_{j+1} are comparable for each j. Since $x < y$ and since B is an antichain, it follows that $x = x_0 < y_0$. Then, since A is an antichain, one has $y_0 > x_1$. Continuing these arguments says that $y_s > x_s = x'$. However, since $y' < x'$, one has $y' < y_s$, which contradicts the fact that B is an antichain.