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Tiling the Boolean lattice with copies of a poset

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Abstract

Let P be a partially ordered set with a unique maximal and minimal element, and size 2^m, where m is a positive integer. Settling a conjecture of Lonc, we prove that if n is sufficiently large, then the Boolean lattice $2^{[n]}$ can be partitioned into isomorphic copies of P. Also, we show that if P has a unique maximum and minimum, but the size of P not necessarily a power of 2, then there exists a constant $c = c(P)$ such that all but at most c elements of $2^{[n]}$ can be covered by disjoint copies of P.

Keywords: Poset tiling, Boolean lattice.

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1 Introduction

The *Boolean lattice* $(2^{[n]}, \subset)$ is the power set of $[n] = \{1, ..., n\}$ ordered by inclusion. If P and Q are partially ordered sets (posets), a subset $P' \subset Q$ is a *copy* of P if the subposet of Q induced on P' is isomorphic to P.

It is an easy exercise to show that every poset P has a copy in $2^{[n]}$ for n sufficiently large. Moreover, if P has a unique maximal and minimal element, then every element of $2^{[n]}$ is contained in some copy of P. Therefore, it is natural to ask whether it is possible to partition $2^{[n]}$ into copies of P.

The case when P is a chain of size 2^k was conjectured by Sands [5]. A slightly more general conjecture was proposed by Griggs [1]: if h is a positive integer and n is sufficiently large, then $2^{[n]}$ can be partitioned into chains such that at most one of the chains in the partition have size different from h. This conjecture was confirmed by Lonc [4]. Moreover, the author of this manuscript [6] gave a different proof of this conjecture and established that the order of the minimal n for which such a partition exists is $\Theta(h^2)$.

Lonc [4] also proposed two conjectures concerning the cases when P is not necessarily a chain. First, he conjectured that if P has a unique maximal and minimal element and size 2^k, then $2^{[n]}$ can be partitioned into copies of P for n sufficiently large. It is easy to see that these conditions on P are necessary. Confirming this conjecture, we show that these conditions are sufficient as well.

Theorem 1.1 *(Gruslys, Leader, Tomon [3]) Let P a poset with a unique maximal and minimal element and $|P| = 2^k$, $k \in \mathbb{N}$. If n is sufficiently large, the Boolean lattice $2^{[n]}$ can be partitioned into copies of P.*

The second conjecture of Lonc targets posets which might not satisfy the conditions stated in Theorem 1.1. In this case, it seems likely that we can still cover almost every element of $2^{[n]}$ with disjoint copies of P.

Conjecture 1.2 *(Lonc [4]) Let P be a poset. If n is sufficiently large and $|P|$ divides $2^n - 2$, then $2^{[n]} \setminus \{\emptyset, [n]\}$ can be partitioned into copies of P.*

This problem proved to be more difficult. In [7], we prove the following weaker version of this conjecture.

Theorem 1.3 *(Tomon [7]) Let P be a poset with a uniqe maximal and minimal element. Then there exists a constant $c = c(P)$ such that for every positive integer n, all but at most c elements of the Boolean lattice $2^{[n]}$ can be covered by disjoint copies of P.*