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# Best possible bounds concerning the set-wise union of families

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ABSTRACT

For two families of sets $\mathcal{F}, \mathcal{G} \subset 2^{[n]}$ we define their set-wise union, $\mathcal{F} \vee \mathcal{G} = \{F \cup G : F \in \mathcal{F}, G \in \mathcal{G}\}$ and establish several – hopefully useful – inequalities concerning $|\mathcal{F} \vee \mathcal{G}|$. Some applications are provided as well.



## 1. Introduction

For a non-negative integer $n$ let $[n] = \{1, \ldots, n\}$ be the standard $n$-element set and $2^{[n]}$ its power set. A subset $\mathcal{F} \subset 2^{[n]}$ is called a *family*. If $G \subset F \in \mathcal{F}$ implies $G \in \mathcal{F}$ for all $G, F \subset [n]$ then $\mathcal{G}$ is called a *complex* (*down-set*). Let $F^c$ denote the complement, $[n] \setminus F$ of $F$. Also let $\mathcal{F}^c = \{F^c : F \in \mathcal{F}\}$ be the *complementary* family. One of the earliest and no doubt the easiest result in extremal set theory, contained in the seminal paper of Erdős, Ko and Rado can be formulated as follows.

**Theorem 0** ([1])**.** *Suppose that there are no $F, G \in \mathcal{F}$ satisfying $F \cup G = [n]$. Then*

$$2 \cdot |\mathcal{F}| \leq 2^n. \tag{1}$$

**Proof.** Just note that the condition implies $\mathcal{F} \cap \mathcal{F}^c = \emptyset$. □

This simple result was the starting point of a lot of research.

**Definition 1.** For a positive integer $t$ let us say that $\mathcal{F} \subset 2^{[n]}$ is *$t$-union* if $|F \cup G| \leq n - t$ for all $F, G \in \mathcal{F}$.

An important result of Katona [3] was the determination of the maximum size of $t$-union families. In the present paper we mostly deal with problems concerning several families.

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**Definition 2.** For positive integers $t$ and $r$, $r \geq 2$ and non-empty families $\mathcal{F}_1, \ldots, \mathcal{F}_r \subset 2^{[n]}$, we say that they are *cross t-union* if $|F_1 \cup \cdots \cup F_r| \leq n - t$ for all $F_1 \in \mathcal{F}_1, \ldots, F_r \in \mathcal{F}_r$.

**Definition 3.** For families $\mathcal{F}, \mathcal{G}$ let $\mathcal{F} \vee \mathcal{G}$ denote their *set-wise union*,

$$\mathcal{F} \vee \mathcal{G} = \{F \cup G : F \in \mathcal{F},\ G \in \mathcal{G}\}.$$

To state our main results we need one more definition. A family $\mathcal{F} \subset 2^{[n]}$ is said to be *covering* if $\{i\} \in \mathcal{F}$ for all $i \in [n]$. If $\mathcal{F}$ is a complex, it is equivalent to saying that $\bigcup_{F \in \mathcal{F}} F = [n]$.

Let us use the term *cross-union* for cross 1-union.

**Theorem 1.** *Suppose that $\mathcal{F}, \mathcal{G} \subset 2^{[n]}$ are cross-union and covering complexes. Then*

$$|\mathcal{F} \vee \mathcal{G}| \geq \frac{7}{8}(|\mathcal{F}| + |\mathcal{G}|). \tag{2}$$

**Example 1.** Let $n \geq 3$ and define $\mathcal{A} = \left\{A \subset [n] : |A \cap [3]| \leq 1\right\}$. Then $|\mathcal{A}| = 2^{n-1}$ and $|\mathcal{A} \vee \mathcal{A}| = \frac{7}{8}2^n$ hold.

The above example shows that (2) is best possible.

**Theorem 2.** *Suppose that $\mathcal{F}, \mathcal{G} \subset 2^{[n]}$ are non-empty cross-union complexes and $\mathcal{F}$ is covering. Then*

$$|\mathcal{F} \vee \mathcal{G}| \geq \frac{3}{4}(|\mathcal{F}| + |\mathcal{G}|). \tag{3}$$

The bound (3) is best possible as shown by the next example.

**Example 2.** Let $n \geq 2$ and define $\mathcal{A} = \{A \subset [n] : |A \cap [2]| \leq 1\}$, $\mathcal{B} = \{B \subset [n] : B \cap [2] = \emptyset\}$.

**Theorem 3.** *Suppose that $\mathcal{F}, \mathcal{G} \subset 2^{[n]}$ are cross 2-union and covering complexes. Then*

$$|\mathcal{F} \vee \mathcal{G}| > |\mathcal{F}| + |\mathcal{G}|. \tag{4}$$

Setting $\mathcal{F} = \{F \subset [n] : |F| \leq n - 3\}$, $\mathcal{G} = \{G \subset [n] : |G| \leq 1\}$ shows that (4) is not far from best possible.

## 2. The proof of Theorems 1 and 2

Let us first note that if $\mathcal{F}, \mathcal{G} \subset 2^{[n]}$ are cross-union then

$$|\mathcal{F}| + |\mathcal{G}| \leq 2^n. \tag{2.1}$$

Indeed the cross-union property guarantees $\mathcal{F} \cap \mathcal{G}^c = \emptyset$ and thereby $|\mathcal{F}| + |\mathcal{G}| = |\mathcal{F}| + |\mathcal{G}^c| \leq |2^{[n]}| = 2^n$.

In view of (2.1) the following statement easily implies Theorem 1.

**Theorem 2.1.** *Let $\mathcal{F}, \mathcal{G} \subset 2^{[n]}$ be covering complexes. Then*

$$|\mathcal{F} \vee \mathcal{G}| \geq \min\left\{2^n, \frac{7}{8}(|\mathcal{F}| + |\mathcal{G}|)\right\}. \tag{2.2}$$

**Proof.** Let us recall the following standard notations. For a family $\mathcal{H} \subset 2^{[n]}$ and $1 \leq i \leq n$,

$$\mathcal{H}(i) = \left\{H \setminus \{i\} : i \in H \in \mathcal{H}\right\} \subset 2^{[n] \setminus \{i\}},$$
$$\mathcal{H}(\bar{i}) = \left\{H \quad : i \notin H \in \mathcal{H}\right\} \subset 2^{[n] \setminus \{i\}}.$$

Note also that for $i \neq j$ if $\{i\} \in \mathcal{H}$ then $\{i\} \in \mathcal{H}(\bar{j})$ as well. In particular, if $\mathcal{H}$ is a covering complex then $\mathcal{H}(\bar{j})$ is a covering complex as well.