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Strong edge-colorings of sparse graphs with large maximum degree

Ilkyoo Choi [a], Jaehoon Kim [b,*], Alexandr V. Kostochka [c,d], André Raspaud [e]

[a] Department of Mathematics, Hankuk University of Foreign Studies, Yongin-si, Gyeonggi-do 17035, Republic of Korea
[b] School of Mathematics, University of Birmingham, Edgbaston, Birmingham, B15 2TT, United Kingdom
[c] University of Illinois at Urbana-Champaign, Urbana, IL 61801, USA
[d] Sobolev Institute of Mathematics, Novosibirsk 630090, Russia
[e] LaBRI (Université de Bordeaux), 351 cours de la Libération, 33405 Talence Cedex, France

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ABSTRACT

A *strong* k*-edge-coloring* of a graph G is a mapping from $E(G)$ to $\{1, 2, \dots, k\}$ such that every two adjacent edges or two edges adjacent to the same edge receive distinct colors. The *strong chromatic index* $\chi'_s(G)$ of a graph G is the smallest integer k such that G admits a strong k-edge-coloring. We give bounds on $\chi'_s(G)$ in terms of the maximum degree $\Delta(G)$ of a graph G when G is sparse, namely, when G is 2-degenerate or when the maximum average degree $\mathrm{Mad}(G)$ is small. We prove that the strong chromatic index of each 2-degenerate graph G is at most $5\Delta(G)+1$. Furthermore, we show that for a graph G, if $\mathrm{Mad}(G) < 8/3$ and $\Delta(G) \geq 9$, then $\chi'_s(G) \leq 3\Delta(G) - 3$ (the bound $3\Delta(G) - 3$ is sharp) and if $\mathrm{Mad}(G) < 3$ and $\Delta(G) \geq 7$, then $\chi'_s(G) \leq 3\Delta(G)$ (the restriction $\mathrm{Mad}(G) < 3$ is sharp).

1. Introduction

A *strong* k*-edge-coloring* of a graph G is a mapping from $E(G)$ to $\{1, 2, \dots, k\}$ such that every two adjacent edges or two edges adjacent to the same edge receive distinct colors. In other words, the

* Corresponding author.
E-mail addresses: ilkyoo@kaist.ac.kr (I. Choi), j.kim.3@bham.ac.uk (J. Kim), kostochk@math.uiuc.edu (A.V. Kostochka), andre.raspaud@labri.fr (A. Raspaud).

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graph induced by each color class is an induced matching. The *strong chromatic index* of G, denoted by $\chi_s'(G)$, is the smallest integer k such that G admits a strong k-edge-coloring.

Strong edge-coloring was introduced by Fouquet and Jolivet [13,14] and was used to solve the frequency assignment problem in some radio networks. For more details on applications see [2,22–24].

An obvious upper bound on $\chi_s'(G)$ (given by a greedy coloring) is $2\Delta(G)(\Delta(G)-1)+1$ where $\Delta(G)$ denotes the maximum degree of G. Erdős and Nešetřil [10,11] conjectured that for every graph G with maximum degree Δ,

$$\chi_s'(G) \leq \begin{cases} \frac{5}{4}\Delta^2 & \text{if } \Delta \text{ is even} \\ \frac{5}{4}\Delta^2 - \frac{\Delta}{2} + \frac{1}{4} & \text{if } \Delta \text{ is odd.} \end{cases}$$

The bounds in the conjecture are sharp, if the conjecture is true.

The first nontrivial upper bound on $\chi_s'(G)$ was given by Molloy and Reed [21], who showed that $\chi_s'(G) \leq 1.998\Delta^2$, if Δ is sufficiently large. The coefficient 1.998 was improved to 1.93 (again, for sufficiently large Δ) by Bruhn and Joos [6]. Recently, Bonamy, Perrett and Postle [4] announced an even better coefficient of 1.835. For $\Delta = 3$, the conjecture was settled independently by Andersen [1] and by Horák, Qing and Trotter [16]. Cranston [9] proved that every graph with $\Delta \leq 4$ admits a strong edge-coloring with 22 colors, which is 2 more than the conjectured bound.

The strong chromatic index was studied for various families of graphs, such as cycles, trees, d-dimensional cubes, chordal graphs, and Kneser graphs, see [20]. There was also a series of papers [12,15,18] on strong edge-coloring planar graphs. In particular, Faudree, Gyárfás, Schelp and Tuza [12] proved that $\chi_s'(G) \leq 4\Delta+4$ for every planar graph G with maximum degree Δ and exhibited, for every integer $\Delta \geq 2$, a planar graph with maximum degree Δ and strong chromatic index $4\Delta - 4$. Borodin and Ivanova [5] showed that every planar graph G with maximum degree $\Delta \geq 3$ and girth $g \geq 40\lfloor \Delta/2 \rfloor$ satisfies $\chi_s'(G) \leq 2\Delta - 1$, and that the bound $2\Delta - 1$ is sharp. Chang, Montassier, Pecher and Raspaud [7] relaxed the restriction on g to $g \geq 10\Delta + 46$ for $\Delta \geq 4$.

Hudák, Lužar, Soták and Škrekovski [17] proved that $\chi_s'(G) \leq 3\Delta+6$ for every planar graph G with maximum degree $\Delta \geq 3$ and girth $g \geq 6$. Recently, Bensmail, Harutyunyan, Hocquard and Valicov [3] improved the upper bound $3\Delta + 6$ for such graphs to $3\Delta + 1$. With the stronger restriction of $g \geq 7$, Ruksasakchai and Wang [25] reduced the bound $3\Delta + 1$ to 3Δ.

Clearly, planar graphs with large girth are sparse. The problem of strong edge-coloring was also studied for general sparse graphs. A natural measure of sparsity is *degeneracy*: a graph G is *d-degenerate* if every subgraph G' of G has a vertex of degree at most k (in G'). Chang and Narayanan [8] proved that $\chi_s'(G) \leq 10\Delta - 10$ for every 2-degenerate graph G with maximum degree $\Delta \geq 2$. Luo and Yu [19] improved the bound $10\Delta - 10$ to $8\Delta - 4$. A more general bound by Yu [27] allowed to reduce the bound for 2-degenerate graphs to $6\Delta - 5$, and Wang [26] improved it to $6\Delta - 7$.

In this paper, we prove three bounds on the strong chromatic index of sparse graphs in terms of the maximum degree. Two of our bounds yield new bounds for planar graphs with girths 6 and 8.

Our first result is on 2-degenerate graphs. It improves the aforementioned bounds in [8,19,26] for $\Delta \geq 9$.

Theorem 1.1. *Every 2-degenerate graph G with maximum degree Δ satisfies $\chi_s'(G) \leq 5\Delta + 1$.*

A finer measure of sparsity is the *maximum average degree*, denoted $\mathrm{Mad}(G)$, which is the maximum of $2\frac{|E(G')|}{|V(G')|}$ over all nontrivial subgraphs G' of a graph G. By definition, $\mathrm{Mad}(G) < 4$ for every 2-degenerate graph G. Two of our results show that if $\mathrm{Mad}(G) < 3$, then we can use significantly fewer than 5Δ colors. The graphs $K_\Delta(t)$ defined below show that our bounds are almost optimal. Let $K_\Delta(t)$ be the graph obtained from K_t by adding $\Delta - t + 1$ pendant edges to each vertex in K_t. It is easy to check that $\mathrm{Mad}(K(t)) = t - 1$ and $\chi_s'(K_\Delta(t)) = |E(K_\Delta(t))| = t\Delta - \binom{t}{2}$. In particular,

- $\mathrm{Mad}(K_\Delta(2)) = 1$ and $\chi_s'(K_\Delta(2)) = 2\Delta - 1$,
- $\mathrm{Mad}(K_\Delta(3)) = 2$ and $\chi_s'(K_\Delta(3)) = 3\Delta - 3$,
- $\mathrm{Mad}(K_\Delta(4)) = 3$ and $\chi_s'(K_\Delta(4)) = 4\Delta - 6$.