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Gorenstein simplices and the associated finite abelian groups

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ABSTRACT

It is known that a lattice simplex of dimension d corresponds a finite abelian subgroup of $(\mathbb{R}/\mathbb{Z})^{d+1}$. Conversely, given a finite abelian subgroup of $(\mathbb{R}/\mathbb{Z})^{d+1}$ such that the sum of all entries of each element is an integer, we can obtain a lattice simplex of dimension d. In this paper, we discuss a characterization of Gorenstein simplices in terms of the associated finite abelian groups. In particular, we present complete characterizations of Gorenstein simplices whose normalized volume equals p, p^2 and pq, where p and q are prime numbers with $p \neq q$. Moreover, we compute the volume of the associated dual reflexive simplices of the Gorenstein simplices.

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0. Introduction

A *lattice polytope* is a convex polytope each of whose vertices has integer coordinates. For a lattice simplex $\Delta \subset \mathbb{R}^d$ of dimension d whose vertices are $v_0, \ldots, v_d \in \mathbb{Z}^d$ set

$$\Lambda_\Delta = \left\{ (\lambda_0, \ldots, \lambda_d) \in (\mathbb{R}/\mathbb{Z})^{d+1} : \sum_{i=0}^{d} \lambda_i (v_i, 1) \in \mathbb{Z}^{d+1} \right\}.$$

The collection Λ_Δ forms a finite abelian group with addition defined as follows: For $(\lambda_0, \ldots, \lambda_d) \in (\mathbb{R}/\mathbb{Z})^{d+1}$ and $(\lambda'_0, \ldots, \lambda'_d) \in (\mathbb{R}/\mathbb{Z})^{d+1}$, $(\lambda_0, \ldots, \lambda_d) + (\lambda'_0, \ldots, \lambda'_d) = (\lambda_0 + \lambda'_0, \ldots, \lambda_d + \lambda'_d) \in (\mathbb{R}/\mathbb{Z})^{d+1}$. Moreover, the order of Λ_Δ equals the *normalized volume* of Δ, i.e., $d!$ times the usual euclidean volume of Δ, which we denote by $\mathrm{Vol}(\Delta)$. Let $\mathbb{Z}^{d\times d}$ be the set of $d \times d$ integral matrices. Recall that a matrix $A \in \mathbb{Z}^{d\times d}$ is *unimodular* if $\det(A) = \pm 1$. Given lattice polytopes $\mathcal{P}$ and $\mathcal{Q}$ in $\mathbb{R}^d$

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of dimension d, we say that $\mathcal{P}$ and $\mathcal{Q}$ are *unimodularly equivalent* if there exist a unimodular matrix $U \in \mathbb{Z}^{d\times d}$ and an integral vector w such that $\mathcal{Q} = f_U(\mathcal{P}) + w$, where f_U is the linear transformation in $\mathbb{R}^d$ defined by U, i.e., $f_U(\mathbf{v}) = \mathbf{v}U$ for all $\mathbf{v} \in \mathbb{R}^d$. In [2], it is shown that there is a bijection between unimodular equivalence classes of d-dimensional lattice simplices with a chosen ordering of their vertices and finite subgroups of $(\mathbb{R}/\mathbb{Z})^{d+1}$ such that the sum of all entries of each element is an integer. In particular, two lattice simplices Δ and Δ' are unimodularly equivalent if and only if there exists an ordering of their vertices such that $\Lambda_\Delta = \Lambda_{\Delta'}$.

A lattice polytope $\mathcal{P} \subset \mathbb{R}^d$ is called *reflexive* if the origin of $\mathbb{R}^d$ is the unique lattice point belonging to the interior of $\mathcal{P}$ and its dual polytope, i.e.,

$$\mathcal{P}^\vee : \{y \in \mathbb{R}^d : \langle x, y\rangle \le 1 \text{ for all } x \in \mathcal{P}\},$$

is also a lattice polytope, where $\langle x, y\rangle$ is the usual inner product of $\mathbb{R}^d$. We say that a lattice polytope $\mathcal{P} \subset \mathbb{R}^d$ is *Gorenstein of index r* where $r \in \mathbb{Z}_{>0}$ if there exist a reflexive polytope $\mathcal{Q} \subset \mathbb{R}^d$ and a lattice point $w \in \mathbb{Z}^d$ such that $\mathcal{Q} = r\mathcal{P} + w$ [11]. Equivalently, the semigroup algebra associated to the cone over $\mathcal{P}$ is a Gorenstein algebra. We call $\mathcal{Q}$ the *associated dual reflexive polytope* of $\mathcal{P}$. Gorenstein polytopes are of interest in combinatorial commutative algebra, mirror symmetry and tropical geometry (for details we refer to [1,8]). For a lattice polytope $\mathcal{P} \subset \mathbb{R}^d$ of dimension d, we can construct a new lattice polytope

$$\mathrm{Pyr}(\mathcal{P}) : \mathrm{conv}(\mathcal{P} \times \{0\}, (0, \dots, 0, 1)) \subset \mathbb{R}^{d+1}$$

of dimension $d + 1$. This polytope $\mathrm{Pyr}(\mathcal{P})$ is called the *lattice pyramid* over $\mathcal{P}$. Then we have $\mathrm{Vol}(\mathcal{P}) = \mathrm{Vol}(\mathrm{Pyr}(\mathcal{P}))$. Moreover, it is known that $\mathcal{P}$ is Gorenstein of index r if and only if $\mathrm{Pyr}(\mathcal{P})$ is Gorenstein of index $r + 1$. Hence, if we construct all Gorenstein polytopes which are not lattice pyramids, we can obtain all Gorenstein polytopes. In each dimension, there exists only finitely many Gorenstein polytopes up to unimodular equivalence [10], and they are known up to dimension 4 [9]. The works [3,7] also provide some classification results for Gorenstein polytopes in the high dimensional setting.

In this paper, we discuss a characterization of Gorenstein simplices in terms of their associated finite abelian groups. In Section 1, we recall the Hermite normal form matrices and some of their properties that we will use in this paper. In Section 2, we prove that a family of simplices arising from Hermite normal form matrices are Gorenstein (Theorem 2.3). Using this result, we characterize Gorenstein simplices whose normalized volume is a prime number. In fact, we will prove the following.

Theorem 0.1. *Let p be a prime number and $\Delta \subset \mathbb{R}^d$ a d-dimensional lattice simplex with normalized volume p. Suppose that Δ is not a lattice pyramid over any lower-dimensional simplex. Then Δ is Gorenstein of index r if and only if $d = rp - 1$ and Λ_Δ is generated by $\left(\frac{1}{p}, \dots, \frac{1}{p}\right)$.*

In Section 3, we extend these results by characterizing Gorenstein simplices whose normalized volume equals p^2 and pq, where p and q are prime numbers with $p \neq q$. In fact, we will prove the following theorems.

Theorem 0.2. *Let p be a prime number and $\Delta \subset \mathbb{R}^d$ a d-dimensional lattice simplex with normalized volume p^2. Suppose that Δ is not a lattice pyramid over any lower-dimensional lattice simplex. Then Δ is Gorenstein of index r if and only if one of the followings is satisfied:*

(1) *There exists an integer s with $0 \le s \le d-1$ such that $rp^2 - 1 = (d - s) + ps$ and Λ_Δ is generated by $\left(\underbrace{\frac{1}{p}, \dots, \frac{1}{p}}_{s}, \underbrace{\frac{1}{p^2}, \dots, \frac{1}{p^2}}_{d-s+1}\right)$ for some ordering of the vertices of Δ.*

(2) *$d = rp - 1$ and there exist an integer s with $1 \le s \le d-1$ and integers $1 \le a_1, \dots, a_{s-1} \le p-1$ such that Λ_Δ is generated by*

$$\left(\frac{2 - \sum_{1\le i\le s-1} a_i}{p}, \frac{a_1 + 1}{p}, \dots, \frac{a_{s-1} + 1}{p}, 0, \frac{1}{p}, \dots, \frac{1}{p}\right)$$

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