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Approximating set multi-covers

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ABSTRACT

Johnson and Lovász and Stein proved independently that any hypergraph satisfies $\tau \leq (1 + \ln \Delta)\tau^*$, where τ is the transversal number, τ^* is its fractional version, and Δ denotes the maximum degree. We prove $\tau_f \leq 3.153\tau^* \max\{\ln \Delta, f\}$ for the f-fold transversal number τ_f. Similarly to Johnson, Lovász and Stein, we also show that this bound can be achieved non-probabilistically, using a greedy algorithm.

As a combinatorial application, we prove an estimate on how fast τ_f/f converges to τ^*. As a geometric application, we obtain an upper bound on the minimal density of an f-fold covering of the d-dimensional Euclidean space by translates of any convex body.

1. Introduction and preliminaries

A *hypergraph* is a pair $(X, \mathcal{F})$, where X is a finite set and $\mathcal{F} \subseteq 2^X$ is a family of some subsets of X. We call the elements of X *vertices*, and the members of $\mathcal{F}$ *edges* of the hypergraph. When a vertex is contained in an edge, we may say that 'the vertex covers the edge', or that 'the edge covers the vertex'.

Let f be a positive integer. An *f-fold transversal* of $(X, \mathcal{F})$ is a multiset A of X such that each member of $\mathcal{F}$ contains at least f elements (with multiplicity). The *f-fold transversal number* τ_f of $(X, \mathcal{F})$ is the minimum cardinality (with multiplicity) of an f-fold transversal. A 1-transversal is called a transversal, and the 1-transversal number is called the transversal number, and is denoted by $\tau = \tau_1$.

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A *fractional transversal* is a function $w : X \longrightarrow [0, 1]$ with $\sum_{x:x\in F} w(x) \geq 1$ for all $F \in \mathcal{F}$. The *fractional transversal number* of $(X, \mathcal{F})$ is

$$\tau^* = \tau^*(\mathcal{F}) := \inf\left\{\sum_{x:x\in X} w(x) \ : \ w \text{ is a fractional transversal}\right\}.$$

Clearly, $\tau^* \leq \tau$. In the opposite direction, Johnson [10], Lovász [11] and Stein [16] independently proved that

$$\tau \leq (1 + \ln \Delta)\tau^*, \tag{1}$$

where Δ denotes the *maximum degree* of $(X, \mathcal{F})$, that is, the maximum number of edges containing a vertex. They showed that the greedy algorithm, that is, picking vertices of X one by one, in such a way that we always pick one that is contained in the largest number of uncovered edges, yields a transversal set whose cardinality does not exceed the right hand side in (1). For more background, see Füredi's survey [9].

Our main result is an extension of this theorem to f-fold transversals.

Theorem 1.1. *Let $\lambda \in (0, 1)$ and let f be a positive integer. Then, with the above notation,*

$$\tau_f \leq \frac{1-\lambda^f}{1-\lambda}\tau^*(1 + \ln \Delta - (f-1)\ln \lambda), \tag{2}$$

moreover, for rational λ, the greedy algorithm using appropriate weights, yields an f-fold transversal of cardinality not exceeding the right hand side of (2).

Substituting $\lambda = 0.287643$ (which is a bit less than $1/e$), we obtain

Corollary 1.2. *With the above notation, we have*

$$\tau_f \leq 3.153\tau^* \max\{\ln \Delta, f\}. \tag{3}$$

This result may be interpreted in two ways. First, it gives an algorithm that approximates the integer programming (IP) problem of finding τ_f, with a better bound on the output of the algorithm than the obvious estimate $\tau_f \leq f\tau \leq f\tau^*(1 + \ln \Delta)$.

A similar result was obtained by Rajagopalan and Vazirani in [14] (an improvement of [3]), where the (multi)-set (multi)-cover problem is considered, that is, the goal is to cover vertices by sets. This is simply the combinatorial dual (and therefore, equivalent) formulation of our problem. In [14], each set can be chosen at most once. They present generalizations of the greedy algorithm of [10,11] and [16], and prove that it finds an approximation of the (multi)-set (multi)-cover problem within an $\ln \Delta$ factor of the optimal solution of the corresponding linear programming (LP) problem. Moreover, they give parallelized versions of the algorithms.

The main difference between [14] and the present paper is that there, the optimal solution of an IP problem is compared to the optimal solution of the LP-relaxation of the same IP problem, whereas here, we compare τ_f with τ^*, where the latter is the optimal solution of a weaker LP problem: the problem with $f = 1$.

We note that, using the fact that $f\tau^* \leq \tau_f$, (3) also implies that the *performance ratio* (that is, the ratio of the value obtained by the algorithm to the optimal value, in the worst case) of our algorithm is constant when $\ln \Delta \leq f$. Compare this with [2, Lemma 1 in Section 3.1], where it is shown that, even for large f, the standard greedy algorithm yields a performance ratio of $\Omega(\ln m)$, where m is the number of sets in the hypergraph. Further recent results on the performance ratio of another modified greedy algorithm for variants of the set cover problem can be found in [8]. See also Chapter 2 of the book [17] by Vazirani.

The second interpretation of our result is the following. It is easy to see that $\frac{\tau_f}{f}$ converges to τ^* as f tends to infinity. Now, (3) quantifies the speed of this convergence in some sense. In particular, it yields that for $f = \ln \Delta$ we have $\frac{\tau_f}{f} \leq 3.153\tau^*$. We have better approximation for larger f.