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Packing and covering odd cycles in cubic plane graphs with small faces[☆]

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ABSTRACT

We show that any 3-connected cubic plane graph on n vertices, with all faces of size at most 6, can be made bipartite by deleting no more than $\sqrt{(p+3t)n/5}$ edges, where p and t are the numbers of pentagonal and triangular faces, respectively. In particular, any such graph can be made bipartite by deleting at most $\sqrt{12n/5}$ edges. This bound is tight, and we characterise the extremal graphs. We deduce tight lower bounds on the size of a maximum cut and a maximum independent set for this class of graphs. This extends and sharpens the results of Faria et al. (2012).

1. Introduction

A set of edges intersecting every odd cycle in a graph is known as an *odd cycle (edge) transversal*, or *odd cycle cover*, and the minimum size of such a set is denoted by τ_{odd}. A set of edge-disjoint odd cycles in a graph is called a *packing of odd cycles*, and the maximum size of such a family is denoted by ν_{odd}. Clearly, $\tau_{\text{odd}} \geq \nu_{\text{odd}}$. Dejter and Neumann-Lara [6] and independently Reed [17] showed that in general, τ_{odd} cannot be bounded by any function of ν_{odd}, i.e., they do not satisfy the Erdős–Pósa property. However, for planar graphs, Král' and Voss [14] proved the (tight) bound $\tau_{\text{odd}} \leq 2\nu_{\text{odd}}$.

In this paper we focus on packing and covering of odd cycles in 3-connected cubic plane graphs with all faces of size at most 6. Such graphs – and their dual triangulations – are a very natural class to consider, as they correspond to surfaces of genus 0 of non-negative curvature (see e.g. [21]).

A much-studied subclass of cubic plane graphs with all faces of size at most 6 is the class of *fullerene graphs*, which only have faces of size 5 and 6. Faria, Klein and Stehlík [9] showed that any fullerene graph on n vertices has an odd cycle transversal with no more than $\sqrt{12n/5}$ edges, and characterised

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the extremal graphs. Our main result is the following extension and sharpening of their result to all 3-connected cubic plane graphs with all faces of size at most 6.

Theorem 1.1. *Let G be a* 3*-connected cubic plane graph on n vertices with all faces of size at most* 6*, with p pentagonal and t triangular faces. Then*

$$\tau_{\mathrm{odd}}(G) \le \sqrt{(p+3t)n/5}.$$

In particular, $\tau_{\mathrm{odd}}(G) \le \sqrt{12n/5}$ *always holds, with equality if and only if all faces have size* 5 *and* 6*,* $n = 60k^2$ *for some* $k \in \mathbb{N}$*, and* $\mathrm{Aut}(G) \cong I_h$*.*

If G is a fullerene graph, then $t = 0$ and Euler's formula implies that $p = 12$, so Theorem 1.1 does indeed generalise the result of Faria, Klein and Stehlík [9]. We also remark that the smallest 3-connected cubic plane graph with all faces of size at most 6 achieving the bound $\tau_{\mathrm{odd}}(G) = \sqrt{12n/5}$ in Theorem 1.1 is the ubiquitous *buckminsterfullerene graph* (on 60 vertices).

The rest of the paper is organised as follows. In Section 2, we introduce the basic notation and terminology, as well as the key concepts from combinatorial optimisation and topology. In Section 3, we introduce the notions of patches and moats, and prove bounds on the area of moats. Then, in Section 4, we use these bounds to prove an upper bound on the maximum size of a packing of T-cuts in triangulations of the sphere with maximum degree at most 6. Using a theorem of Seymour [19], we deduce, in Section 5, an upper bound on the minimum size of a T-join in triangulations of the sphere with maximum degree at most 6, and then dualise to complete the proof of Theorem 1.1. In Section 6, we deduce lower bounds on the size of a maximum cut and a maximum independent set in 3-connected cubic plane graphs with no faces of size more than 6. Finally, in Section 7, we show why the condition on the face size cannot be relaxed, and briefly discuss the special case when the graph contains no pentagonal faces.

2. Preliminaries

Most of our graph-theoretic terminology is standard and follows [1]. All graphs are finite and simple, i.e., have no loops and parallel edges. The *degree* of a vertex u in a graph G is denoted by $d_G(u)$. If all vertices in G have degree 3, then G is a *cubic graph*. The set of edges in G with exactly one end vertex in X is denoted by $\delta_G(X)$. A set C of edges is a *cut* of G if $C = \delta_G(X)$, for some $X \subseteq V(G)$. When there is no risk of ambiguity, we may omit the subscripts in the above notation.

The set of all automorphisms of a graph G forms a group, known as the *automorphism group* $\mathrm{Aut}(G)$. The *full icosahedral group* $I_h \cong A_5 \times C_2$ is the group of all symmetries (including reflections) of the regular icosahedron. The *full tetrahedral group* $T_d \cong S_4$ is the group of all symmetries (including reflections) of the regular tetrahedron.

A *polygonal surface* K is a simply connected 2-manifold, possibly with a boundary, which is obtained from a finite collection of disjoint simple polygons in $\mathbb{R}^2$ by identifying them along edges of equal length. We denote by $|K|$ the union of all polygons in K, and remark that $|K|$ is a surface.

Based on this construction, K may be viewed as a graph *embedded* in the surface $|K|$. Accordingly, we denote its set of vertices, edges, and faces by $V(K)$, $E(K)$, and $F(K)$, respectively. If every face of K is incident to three edges, K is a *triangulated surface*, or a *triangulation* of $|K|$. In this case, K can be viewed as a *simplicial complex*. If K is a simplicial complex and $X \subseteq V(K)$, then $K[X]$ is the subcomplex induced by X, and $K \setminus X$ is the subcomplex obtained by deleting X and all incident simplices. If L is a subcomplex of K, then we simply write $K \setminus L$ instead of $K \setminus V(L)$.

If K is a graph embedded in a surface $|K|$ without boundary, the *dual graph* K^* is the graph with vertex set $F(K)$, such that $fg \in E(K^*)$ if and only if f and g share an edge in K. The size of a face $f \in F(K)$ is defined as the number of edges on its boundary walk, and is denoted by $d_K(f)$. Note that $d_K(f) = d_{K^*}(f^*)$.

Any polygonal surface homeomorphic to a sphere corresponds to a plane graph via the stereographic projection. Therefore, terms such as 'plane triangulation' and 'triangulation of the sphere' can be used interchangeably. We shall make the convention to use the term 'cubic plane graphs' because it is so widespread, but refer to the dual graphs as 'triangulations of the sphere' because it reflects better our geometric viewpoint.