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The serpent nest conjecture for accordion complexes

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ABSTRACT

Consider $2n$ points on the unit circle and a reference dissection $D_\circ$ of the convex hull of the odd points. The accordion complex of $D_\circ$ is the simplicial complex of subsets of pairwise noncrossing diagonals with even endpoints that cross a connected set of diagonals of the dissection $D_\circ$. In particular, this complex is an associahedron when $D_\circ$ is a triangulation, and a Stokes complex when $D_\circ$ is a quadrangulation. We exhibit a bijection between the facets of the accordion complex of $D_\circ$ and some dual objects called the serpent nests of $D_\circ$. This confirms in particular a prediction of F. Chapoton (2016) in the case of Stokes complexes.

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1. Introduction

1.1. Motivations

Y. Baryshnikov introduced in [1] the definition of a *Stokes complex*, namely the simplicial complex of dissections of a polygon that are in some sense compatible with a reference quadrangulation $Q_\circ$. Although the precise definition of compatibility is a bit technical in [1], it turns out that a diagonal is compatible with $Q_\circ$ if and only if it crosses a connected subset of diagonals of a slightly rotated version of $Q_\circ$, that we call an *accordion* of $Q_\circ$. We therefore also call Y. Baryshnikov's simplicial complex the *accordion complex* $\mathcal{AC}(Q_\circ)$ of $Q_\circ$. As an example, this complex coincides with the classical associahedron when all the diagonals of the reference quadrangulation $Q_\circ$ have a common endpoint. Revisiting some combinatorial and algebraic properties of $\mathcal{AC}(Q_\circ)$, F. Chapoton [2] raised three challenges: first prove that the dual graph of $\mathcal{AC}(Q_\circ)$, suitably oriented, has a lattice structure extending the Tamari and Cambrian lattices [6–8]; second construct geometric realizations of $\mathcal{AC}(Q_\circ)$ as fans and polytopes

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generalizing the known constructions of the associahedron; third show enumerative properties of the faces of $\mathcal{AC}(Q_\circ)$, among which he expects a bijection to exist between the facets of $\mathcal{AC}(Q_\circ)$ and other combinatorial objects called *serpent nests*. These three challenges are evoked in the introduction of [2], respectively, at paragraph 22, last paragraph and paragraph 15. The serpent nest conjecture is also a specialization of [2, Conjecture 4.5] for $x = y = 1$. Serpent nests are essentially special sets of dual paths in the dual tree of the reference quadrangulation $Q_\circ$. As for the two other challenges, their study is related to extensions of known phenomena on the associahedron. Serpent nests are indeed expected by F. Chapoton to play the same role towards Stokes complexes as nonnesting partitions towards associahedra. The serpent nest conjecture therefore morally asserts that the fact that nonnesting partitions are in bijection with triangulations of convex polygons holds in the more general context of Stokes complexes.

In [3], A. Garver and T. McConville defined and studied the accordion complex $\mathcal{AC}(D_\circ)$ of any reference dissection $D_\circ$. Our presentation slightly differs from their's as they use a compatibility condition on the dual tree of the dissection $D_\circ$, but the simplicial complex is the same. In this context, they settled F. Chapoton's lattice question, using lattice quotients of a lattice of biclosed sets. In a paper of T. Manneville and V. Pilaud [5], geometric realizations (as fans and convex polytopes) of $\mathcal{AC}(D_\circ)$ were given for any reference dissection $D_\circ$, providing in particular an answer to F. Chapoton's geometric question. The present paper settles the serpent nest conjecture of F. Chapoton, in a version extended to any accordion complex. Other enumerative conjectures involving a statistic called *F-triangle* are proposed in [2]. A proof that this statistic is preserved by the *twist* operation [2, Conjecture 2.6] can be found in [4, Section 8.3.2], but this result should go together with others that remain open for the moment.

1.2. Overview

Section 2 introduces the accordion complex of a dissection $D_\circ$. We follow the presentation already adopted in [5], where the definitions and arguments of A. Garver and T. McConville [3] are adapted to work directly on the dissection $D_\circ$ rather than on its dual graph. We define serpent nests in Section 3 and present there our bijection between the facets of $\mathcal{AC}(D_\circ)$ and the serpent nests of $D_\circ$.

2. Accordion dissections

By a *diagonal* of a convex polygon $\mathcal{P}$, we mean either an internal diagonal or an external diagonal (boundary edge) of $\mathcal{P}$, but a *dissection* D of $\mathcal{P}$ is a set of pairwise noncrossing *internal* diagonals of $\mathcal{P}$. We denote diagonals as pairs (i, j) of vertices, with $i \leq j$ when the labels on vertices are ordered. We moreover denote by $\bar{D}$ the dissection D together with all boundary edges of $\mathcal{P}$. The *cells* of D are the bounded connected components of the plane minus the diagonals of $\bar{D}$. An *accordion* of D is a subset of $\bar{D}$ which contains either no or two incident diagonals in each cell of D. A *subaccordion* of D is a subset of D formed by the diagonals between two given internal diagonals in an accordion of D. A *zigzag* of D is a subset $\{\delta_0, \ldots, \delta_{p+1}\}$ of D where δ_i shares distinct endpoints with δ_{i-1} and δ_{i+1} and separates them for any $i \in [p]$. The *zigzag* of an accordion A is the subset of the diagonals of A which disconnect A. Notice that accordions of D contain boundary edges of $\mathcal{P}$, but not subaccordions nor zigzags. See Fig. 1 for illustrations.

Consider $2n$ points on the unit circle labeled clockwise by $1_\circ, 2_\bullet, 3_\circ, 4_\bullet, \ldots, (2n-1)_\circ, (2n)_\bullet$ (with labels meant modulo $2n$). We say that $1_\circ, \ldots, (2n-1)_\circ$ are the *hollow vertices* while $2_\bullet, \ldots, (2n)_\bullet$ are the *solid vertices*. The *hollow polygon* is the convex hull $\mathcal{P}_\circ$ of $1_\circ, \ldots, (2n-1)_\circ$ while the *solid polygon* is the convex hull $\mathcal{P}_\bullet$ of $2_\bullet, \ldots, (2n)_\bullet$. We simultaneously consider *hollow diagonals* $\delta_\circ$ (with two hollow vertices) and *solid diagonals* $\delta_\bullet$ (with two solid vertices), but never consider diagonals with vertices of each kind. Similarly, we consider *hollow dissections* $D_\circ$ (with only hollow diagonals) and *solid dissections* $D_\bullet$ (with only solid diagonals), but never mix hollow and solid diagonals in a dissection. To distinguish them more easily, hollow (resp. solid) vertices and diagonals appear red (resp. blue) in all pictures.

Let $D_\circ$ be an arbitrary reference hollow dissection. A *$D_\circ$-accordion diagonal* is a solid diagonal $\delta_\bullet$ such that the hollow diagonals of $\bar{D}_\circ$ crossed by $\delta_\bullet$ form an accordion of $D_\circ$. In other words, $\delta_\bullet$

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