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Bootstrap percolation, and other automata

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ABSTRACT

Many fundamental and important questions from statistical physics lead to beautiful problems in extremal and probabilistic combinatorics. One particular example of this phenomenon is the study of bootstrap percolation, which is motivated by a variety of 'real-world' cellular automata, such as the Glauber dynamics of the Ising model of ferromagnetism, and kinetically constrained spin models of the liquid–glass transition.

In this review article, we will describe some dramatic recent developments in the theory of bootstrap percolation (and, more generally, of monotone cellular automata with random initial conditions), and discuss some potential extensions of these methods and results to other automata. In particular, we will state numerous conjectures and open problems.

1. Introduction

Cellular automata are interacting particle systems whose update rules are 'local' and homogeneous. In recent years, a great deal of progress has been made in understanding the behaviour of a particular class of *monotone* cellular automata, commonly known as 'bootstrap percolation'. In particular, if one considers only two-dimensional automata, then we now have a fairly precise understanding of the typical evolution of these processes, starting from random initial conditions. The aim of this article is to describe some of these developments, and discuss a number of open problems and conjectures about related cellular automata. In particular, we will focus our attention on kinetically constrained spin models, the Glauber dynamics of the zero-temperature Ising model, and the abelian sandpile.[1]

Let us begin by defining a large class of d-dimensional monotone cellular automata, which were recently introduced by Bollobás, Smith and Uzzell [14].

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[1] We would like to reassure the reader that the definitions of these models are all purely combinatorial, non-technical, and easy to understand, and that no knowledge of physics will be required in this paper.

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Definition 1.1. Let $\mathcal{U} = \{X_1, \dots, X_m\}$ be an arbitrary finite collection of finite subsets of $\mathbb{Z}^d \setminus \{\mathbf{0}\}$. The *$\mathcal{U}$-bootstrap process* on the d-dimensional torus $\mathbb{Z}_n^d$ is defined as follows: given a set $A \subset \mathbb{Z}_n^d$ of initially *infected* sites, set $A_0 = A$, and define for each $t \geqslant 0$,

$$A_{t+1} = A_t \cup \left\{ v \in \mathbb{Z}_n^d \,:\, v + X \subset A_t \text{ for some } X \in \mathcal{U} \right\}.$$

We write $[A]_{\mathcal{U}} = \bigcup_{t \geqslant 0} A_t$ for the *closure* of A under the $\mathcal{U}$-bootstrap process.

Thus, a vertex v becomes infected at time $t+1$ if the translate by v of one of the sets in $\mathcal{U}$ (which we refer to as the *update family*) is already entirely infected at time t, and infected vertices remain infected forever. For example, if we take $\mathcal{U}$ to be $\mathcal{N}_r^d$, the family of r-subsets of the $2d$ nearest neighbours in $\mathbb{Z}^d$ of the origin, we obtain the classical r-neighbour bootstrap process, which was first introduced in 1979 by Chalupa, Leath and Reich [19].

We are interested in the typical global behaviour of the $\mathcal{U}$-bootstrap process acting on random initial sets. One of the key insights of Bollobás, Smith and Uzzell [14] was that, at least in two dimensions, this behaviour should be determined by the action of the process on discrete half-planes. To be more precise, for each $u \in S^1$, let $\mathbb{H}_u := \{x \in \mathbb{Z}^2 : \langle x, u \rangle < 0\}$ be the discrete half-plane whose boundary is perpendicular to u. We say that u is a *stable direction* if $[\mathbb{H}_u]_{\mathcal{U}} = \mathbb{H}_u$ and we denote by $\mathcal{S} = \mathcal{S}(\mathcal{U}) \subset S^1$ the collection of stable directions. Let us say that a two-dimensional update family $\mathcal{U}$ is:

- *supercritical* if there exists an open semicircle in S^1 that is disjoint from $\mathcal{S}$,
- *critical* if there exists a semicircle in S^1 that has finite intersection with $\mathcal{S}$, and if every open semicircle in S^1 has non-empty intersection with $\mathcal{S}$,
- *subcritical* if every semicircle in S^1 has infinite intersection with $\mathcal{S}$.

To justify this trichotomy, we need a couple more simple definitions. Let us say that a set $A \subset \mathbb{Z}_n^d$ is *p-random* if each of the vertices of $\mathbb{Z}_n^d$ is included in A independently with probability p, and define the *critical probability* of the $\mathcal{U}$-bootstrap process on $\mathbb{Z}_n^d$ to be

$$p_c\big(\mathbb{Z}_n^d, \mathcal{U}\big) := \inf\Big\{ p \,:\, \mathbb{P}_p\big([A]_{\mathcal{U}} = \mathbb{Z}_n^d\big) \geqslant 1/2 \Big\},$$

where $\mathbb{P}_p$ denotes the product probability measure on $\mathbb{Z}_n^d$ with density p.[2] The following theorem was proved by Bollobás, Smith and Uzzell [14] (parts (a) and (b)) and by Balister, Bollobás, Przykucki and Smith [6] (part (c)).

Theorem 1.2. *Let $\mathcal{U}$ be a two-dimensional update family.*

(a) *If $\mathcal{U}$ is supercritical then* $p_c(\mathbb{Z}_n^2, \mathcal{U}) = n^{-\Theta(1)}$.
(b) *If $\mathcal{U}$ is critical then* $p_c(\mathbb{Z}_n^2, \mathcal{U}) = (\log n)^{-\Theta(1)}$.
(c) *If $\mathcal{U}$ is subcritical then* $\liminf_{n \to \infty} p_c(\mathbb{Z}_n^2, \mathcal{U}) > 0$.

It is perhaps difficult to convey to the reader how surprising it is that such a simple and beautiful characterization could be proved in such extraordinary generality. Nevertheless, this is not the end of the story: to state the main result of [12], which determines $p_c(\mathbb{Z}_n^2, \mathcal{U})$ for critical families up to a *constant* factor, we will need a couple more definitions.

Let $\mathbb{Q}_1 \subset S^1$ denote the set of rational directions on the circle, and for each $u \in \mathbb{Q}_1$, let ℓ_u^+ be the (infinite) subset of the line $\ell_u := \{x \in \mathbb{Z}^2 : \langle x, u \rangle = 0\}$ consisting of the origin and the sites to the right of the origin as one looks in the direction of u. Similarly, let $\ell_u^- := (\ell_u \setminus \ell_u^+) \cup \{\mathbf{0}\}$ consist of the origin and the sites to the left of the origin.

Now, given a two-dimensional bootstrap percolation update family $\mathcal{U}$, let $\alpha_{\mathcal{U}}^+(u)$ be the minimum (possibly infinite) cardinality of a set $Z \subset \mathbb{Z}^2$ such that $[\mathbb{H}_u \cup Z]_{\mathcal{U}}$ contains infinitely many sites of ℓ_u^+, and define $\alpha_{\mathcal{U}}^-(u)$ similarly (using ℓ_u^- in place of ℓ_u^+).

2 Thus a p-random set is one chosen according to the distribution $\mathbb{P}_p$.

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