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Chromatic number of random Kneser hypergraphs

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ABSTRACT

Recently, Kupavskii (2016) [20] investigated the chromatic number of random Kneser graphs $\mathrm{KG}_{n,k}(\rho)$ and proved that, in many cases, the chromatic number of the random Kneser graph $\mathrm{KG}_{n,k}(\rho)$ and the Kneser graph $\mathrm{KG}_{n,k}$ are almost surely closed. He also marked the studying of the chromatic number of random Kneser hypergraphs $\mathrm{KG}^r_{n,k}(\rho)$ as a very interesting problem. With the help of $\mathbb{Z}_p$-Tucker lemma, a combinatorial generalization of the Borsuk–Ulam theorem, we generalize Kupavskii's result to random general Kneser hypergraphs by introducing an almost surely lower bound for the chromatic number of them. Roughly speaking, as a special case of our result, we show that the chromatic numbers of the random Kneser hypergraph $\mathrm{KG}^r_{n,k}(\rho)$ and the Kneser hypergraph $\mathrm{KG}^r_{n,k}$ are almost surely closed in many cases. Moreover, restricting to the Kneser and Schrijver graphs, we present a purely combinatorial proof for an improvement of Kupavskii's result.

Also, for any hypergraph $\mathcal{H}$, we present a lower bound for the minimum number of colors required in a coloring of $\mathrm{KG}^r(\mathcal{H})$ with no monochromatic $K^r_{t,\ldots,t}$ subhypergraph, where $K^r_{t,\ldots,t}$ is the complete r-uniform r-partite hypergraph with tr vertices such that each of its parts has t vertices. This

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result generalizes the lower bound for the chromatic number of $\mathrm{KG}^r(\mathcal{H})$ found by Alishahi and Hajiabolhassan (2015) [3].

1. Introduction and main results

For positive integers n and k, by the symbols $[n]$ and $\binom{[n]}{k}$, we mean the set $\{1,\ldots,n\}$ and the set of all k-subsets of $[n]$, respectively. A hypergraph $\mathcal{H}$ is a pair $(V(\mathcal{H}),E(\mathcal{H}))$, where $V(\mathcal{H})$ is a finite nonempty set and $E(\mathcal{H})$ is a family of distinct nonempty subsets of $V(\mathcal{H})$. Respectively, the sets $V(\mathcal{H})$ and $E(\mathcal{H})$ are called the vertex set and the edge set of $\mathcal{H}$. For a subset $M\subseteq V(\mathcal{H})$, the induced subhypergraph by the vertices in M, denoted by $\mathcal{H}[M]$, is a hypergraph with the vertex set M and the edge set $\left\{e\in E(\mathcal{H})\colon e\subseteq M\right\}$. If each edge of $\mathcal{H}$ has the cardinality r, then $\mathcal{H}$ is called r-uniform. A 2-uniform hypergraph is simply called a graph. Let $\mathcal{H}$ be an r-uniform hypergraph and $V_1,\ldots,V_r$ be pairwise disjoint subsets of $V(\mathcal{H})$. The hypergraph $\mathcal{H}[V_1,\ldots,V_r]$ is a subhypergraph of $\mathcal{H}$ whose vertex set and edge set are respectively $\bigcup\limits_{i=1}^{r}V_i$ and

$$E(\mathcal{H}[V_1,\ldots,V_r])=\left\{e\in E(\mathcal{H}):\ e\subseteq\bigcup_{i=1}^{r}V_i \text{ and } |e\cap V_i|=1 \text{ for each } i\in[r]\right\}.$$

For a positive integer $r\geq 2$, *the Kneser hypergraph* $\mathrm{KG}^r_{n,k}$ is a hypergraph which has the vertex set $\binom{[n]}{k}$, and whose edges are formed by the r-sets $\{e_1,\ldots,e_r\}$ where $e_1,\ldots,e_r$ are pairwise disjoint members of $\binom{[n]}{k}$. Kneser (1955) [18] conjectured that for $n\geq 2k$, the chromatic number of $\mathrm{KG}^2_{n,k}$ is $n-2k+2$. After more than 20 years, in a fascinating paper, Lovász [21] gave an affirmative answer to Kneser's conjecture using algebraic topology. Lovász's paper is known as the beginning of the study of combinatorial problems by using topological tools, which is called topological combinatorics. Later, in 1986, Alon, Frankl and Lovász [7] generalized Lovász's result to Kneser hypergraphs by proving that for $n\geq rk$,

$$\chi(\mathrm{KG}^r_{n,k})=\left\lceil\frac{n-r(k-1)}{r-1}\right\rceil.$$

This result also gave a positive answer to a conjecture posed by Erdős [15]. Schrijver [24] improved Lovász's result by introducing a subgraph $\mathrm{SG}_{n,k}$ of $\mathrm{KG}^2_{n,k}$, called the Schrijver graph, which is a vertex critical graph having the same chromatic number as that of $\mathrm{KG}^2_{n,k}$. A *stable subset of* $[n]$ is a set $A\subseteq[n]$ such that for each $i\neq j\in A$, we have $2\leq|i-j|\leq n-2$. Let $\binom{[n]}{k}_{stable}$ be the set of all stable k-subsets of $[n]$. *The Schrijver graph* $\mathrm{SG}_{n,k}$ is the induced subgraph of $\mathrm{KG}^2_{n,k}$ by the vertices in $\binom{[n]}{k}_{stable}$.