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Structure of group invariant weighing matrices of small weight

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ABSTRACT

We show that every weighing matrix of weight n invariant under a finite abelian group G can be generated from a subgroup H of G with $|H| \leq 2^{n-1}$. Furthermore, if n is an odd prime power and a proper circulant weighing matrix of weight n and order v exists, then $v \leq 2^{n-1}$. We also obtain a lower bound on the weight of group invariant matrices depending on the invariant factors of the underlying group. These results are obtained by investigating the structure of subsets of finite abelian groups that do not have unique differences.

1. Introduction

Let G be a finite multiplicative group of order v and let $\mathbb{Z}[G]$ denote the corresponding integral group ring. Any $X \in \mathbb{Z}[G]$ can be written as $X = \sum_{g\in G} a_g g$ with $a_g \in \mathbb{Z}$. The integers a_g are the **coefficients** of X. We write $|X| = \sum_{g\in G} a_g$ and $X^{(-1)} = \sum a_g g^{-1}$. We identify a subset S of G with the group ring element $\sum_{g\in S} g$. For the identity

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element 1_G of G and an integer s, we write s for the group ring element $s1_G$. The set $\mathrm{supp}(X) = \{g \in G : a_g \neq 0\}$ is called the **support** of X.

A **weighing matrix of order *v*** is a $v \times v$ matrix M with entries $0, \pm 1$ only such that $MM^T = nI$ where n is a positive integer and I is an identity matrix. The integer n is the **weight** of the matrix. Let G be a finite group and let $H = (h_{f,g})_{f,g \in G}$ be a $|G| \times |G|$ matrix, indexed with the elements of G. We say that H is ***G*-invariant** if $h_{fk,gk} = h_{f,g}$ for all $f, g, k \in G$. Weighing matrices invariant under cyclic groups are called **circulant weighing matrices**.

The existence of group invariant weighing matrices has been studied quite intensively, see [1–7,9–12,17,18], for instance. Interest in methods for the study of group invariant weighing matrices also stems from the multiplier conjecture for difference sets: The most powerful known approach to this conjecture due to McFarland [13] depends on nonexistence results for group ring elements which satisfy the same equation $XX^{(-1)} = n$ as group invariant weighing matrices. In fact, our results can be used to improve multiplier theorems.

The following is well known, see [16, Lemma 1.3.9].

Lemma 1. *Let G be a finite group of order v. The existence of G-invariant weighing matrix of weight n is equivalent to the existence of $X \in \mathbb{Z}[G]$ with coefficients $0, \pm 1$ only such that $XX^{(-1)} = n$.*

In view of Lemma 1, we will always view G-invariant weighing matrices as elements of $\mathbb{Z}[G]$. The key to our results is the investigation of the support of group invariant weighing matrices in Section 3. As the support of such matrices does not contain a unique difference, we can use the Smith Normal Form of the matrix of the corresponding linear system to gain insights into the structure of the support.

Many group invariant weighing matrices can be constructed as follows. Let H be a subgroup of a finite abelian group G and let $g_1, \ldots, g_K \in G$ be representatives of distinct cosets of H in G. Suppose that $X_1, \ldots, X_K \in \mathbb{Z}[H]$ have coefficients $0, \pm 1$ only and that $\sum_{i=1}^K X_i X_i^{(-1)} = n$ and $X_i X_j = 0$ whenever $i \neq j$. It follows by straightforward computation ([2, Theorem 2.4]) that

$$X = \sum_{i=1}^{K} X_i g_i \tag{1}$$

is a G-invariant weighing matrix of weight n. If (1) holds, we say that X is **generated from *H***.

Note that, indeed, the main conditions that make (1) a weighing matrix only involve equations over the group ring of H. These conditions are $\sum_{i=1}^K X_i X_i^{(-1)} = n$ and $X_i X_j = 0$ for $i \neq j$. The choice of the g_i's only makes sure that the coefficients of X are $0, \pm 1$. In fact, such g_i's exist in *any* abelian group which contains H as a subgroup of index at least K.