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Ordered set partition statistics and the Delta Conjecture

Brendon Rhoades

Department of Mathematics, University of California, San Diego, La Jolla, CA, 92093-0112, USA

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ABSTRACT

The Delta Conjecture of Haglund, Remmel, and Wilson is a recent generalization of the Shuffle Conjecture in the field of diagonal harmonics. In this paper we give evidence for the Delta Conjecture by proving a pair of conjectures of Wilson and Haglund–Remmel–Wilson which give equidistribution results for statistics related to inversion count and major index on objects related to ordered set partitions. Our results generalize the famous result of MacMahon that major index and inversion number share the same distribution on permutations.

1. Introduction

The recently proven Shuffle Conjecture asserts a formula for the bigraded Frobenius character of the module of diagonal harmonics in terms of a Macdonald eigenoperator and the combinatorics of parking functions [6,10]. The Delta Conjecture is a still-open generalization of the Shuffle Conjecture due to Haglund, Remmel, and Wilson [12]. This paper proves combinatorial results about statistics on ordered set partitions to obtain a

E-mail address: bprhoades@math.ucsd.edu.

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specialization of the Delta Conjecture. We begin by reviewing the Delta Conjecture and then turn our attention to combinatorics (where it will remain for the rest of the paper).

1.1. The Delta Conjecture

Let Λ denote the ring of symmetric functions in the variable set $x = (x_1, x_2, \dots)$ with coefficients in the field $\mathbb{Q}(q,t)$. For any partition μ, let $\tilde{H}_\mu = \tilde{H}_\mu(x;q,t) \in \Lambda$ denote the corresponding modified Macdonald symmetric function; it is known that

$$\{\tilde{H}_\mu \,:\, \mu \text{ a partition}\}$$

is a basis for Λ.

For any symmetric function $f = f(x_1, x_2, \dots) \in \Lambda$, define the *Delta operator* $\Delta'_f : \Lambda \to \Lambda$ to be the Macdonald eigenoperator given by

$$\Delta'_f : \tilde{H}_\mu \mapsto f[B_\mu(q,t) - 1]\tilde{H}_\mu. \tag{1.1}$$

Here we are using the plethystic shorthand $f[B_\mu(q,t) - 1] = f(\dots, q^{i-1}t^{j-1}, \dots)$, where (i,j) ranges over the matrix coordinates of all cells $\neq (1,1)$ in the (English) Young diagram of μ.

For example, suppose $\mu = (3,2,1) \vdash 6$ is the partition whose Young diagram is shown below. Then

$$\Delta'_f : \tilde{H}_{(3,2,1)} \mapsto f[B_{(3,2,1)}(q,t) - 1]\tilde{H}_{(3,2,1)} = f(q, q^2, t, qt, t^2)\tilde{H}_{(3,2,1)},$$

where we implicitly set $x_n = 0$ in $f(x_1, x_2, \dots)$ for all $n > 5$.

$\bullet$	q	q^2
t	qt	
t^2		

Let $e_d \in \Lambda$ denote the elementary symmetric function of degree d. The Delta Conjecture ([12, Conjecture 1.1]) asserts a combinatorial formula for the application of Δ'_{e_k} to e_n for all nonnegative integers $k < n$.

Conjecture 1.1. (The Delta Conjecture) *For any integers $n > k \geq 0$,*

$$\Delta'_{e_k} e_n = \{z^{n-k-1}\} \left[\sum_{P \in \mathcal{LD}_n} q^{\mathrm{dinv}(P)} t^{\mathrm{area}(P)} \prod_{i : a_i(P) > a_{i-1}(P)} \left(1 + z/t^{a_i(P)}\right) x^P \right] \tag{1.2}$$

$$= \{z^{n-k-1}\} \left[\sum_{P \in \mathcal{LD}_n} q^{\mathrm{dinv}(P)} t^{\mathrm{area}(P)} \prod_{i \in \mathrm{Val}(P)} \left(1 + z/q^{d_i(P)+1}\right) x^P \right]. \tag{1.3}$$

Here the operator $\{z^{n-k-1}\}$ extracts the coefficient of z^{n-k-1}.