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Decomposing almost complete graphs by random trees

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ABSTRACT

An old conjecture of Ringel states that every tree with m edges decomposes the complete graph K_{2m+1}. The best known lower bound for the order of a complete graph which admits a decomposition by every given tree with m edges is $O(m^3)$. We show that asymptotically almost surely a random tree with m edges and $p = 2m+1$ a prime decomposes $K_{2m+1}(r)$ for every $r \geq 2$, the graph obtained from the complete graph K_{2m+1} by replacing each vertex by a coclique of order r. Based on this result we show, among other results, that a random tree with $m+1$ edges a.a.s. decomposes the compete graph K_{6m+5} minus one edge.

1. Introduction

Given two graphs H and G we say that H decomposes G if G is the edge-disjoint union of isomorphic copies of H. The following is a well-known conjecture of Ringel.

Conjecture 1 *(Ringel [17]). Every tree with m edges decomposes the complete graph K_{2m+1}.*

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The conjecture has been verified by a number of particular classes of trees, see the extensive survey by Gallian [10]. Recently, there have been substantial contributions in the area of graph decomposition problems which are partly motivated by Conjecture 1 leading to impressive results. Böttcher, Hladký, Piguet and Taraz [3] show that, for any $\epsilon > 0$ and any Δ, every family of trees with orders at most n maximum degree at most Δ and at most $\binom{n}{2}$ edges in total packs into the complete graph $K_{(1+\epsilon)n}$ for every sufficiently large n. This provides an approximate result to several tree packing conjectures including Conjecture 1. The result was extended by Mesuti, Rödl and Schacht [16] from trees to general minor closed classes of bounded degree graphs and by Ferber, Lee and Moussat [9] to separable classes of bounded degree graphs. Recently, Joos, Kim, Kühn and Osthus [12] gave a general result on packing trees which in particular definitely proves Conjecture 1 for bounded degree trees and sufficiently large m.

In this paper we aim at results for almost all trees. By a random tree of size m we mean an unlabelled tree chosen uniformly at random among all the unlabelled trees with m edges. Drmota and the author [8] used structural results on random trees to show that asymptotically almost surely (a.a.s.) a random tree with m edges decomposes the complete bipartite graph $K_{2m,2m}$, thus providing an approximate result to another decomposition conjecture by Graham and Häggkvist (see e.g. [11]) which asserts that the complete bipartite graph $K_{m,m}$ can be decomposed by any given tree with m edges.

Let $g(m)$ be the smallest integer n such that any tree with m edges decomposes the complete graph K_n. It was shown by Yuster [20] that $g(m) = O(m^{10})$ and the upper bound was reduced by Kezdy and Snevily [14] to $g(m) = O(m^3)$. Since $K_{2m,2m}$ decomposes the complete graph K_{8m^2+1} (see Snevily [19]), the above mentioned result on the decomposition of $K_{2m,2m}$ shows that a tree with m edges decomposes a.a.s. the complete graph K_{8m^2+1}, giving a quadratic bound on m for almost all trees. Our aim is to reduce this quadratic bound to linear getting much closer to Ringel's conjecture.

For positive integers n, r we denote by $K_n(r)$ the blow-up graph obtained from the complete graph K_n by replacing each vertex by a coclique of order r (r mutually nonadjacent vertices) and joining every pair of vertices that do not belong to the same coclique. Our main result is the following one.

Theorem 1. *For every $r \geq 2$, asymptotically almost surely a random tree with m edges such that $p = 2m + 1$ is a prime decomposes $K_{2m+1}(r)$.*

Theorem 1 will be derived from the following deterministic result which considers trees with sufficiently many leaves.

Theorem 2. *Let $p > 10$ be a prime and $r \geq 2$ an integer. Let T be a tree with $m = (p-1)/2$ edges. If T has at least $2m/5$ leaves, then T decomposes $K_{2m+1}(r)$.*