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Tremain equiangular tight frames ☆

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A B S T R A C T

Equiangular tight frames provide optimal packings of lines through the origin. We combine Steiner triple systems with Hadamard matrices to produce a new infinite family of equiangular tight frames. This in turn leads to new constructions of strongly regular graphs and distance-regular antipodal covers of the complete graph.

1. Introduction

Consider the following problem: Given a finite-dimensional Hilbert space $\mathcal{H}$ and some positive integer N, find vectors $\varphi_1, \ldots, \varphi_N \in \mathcal{H}$ of unit length that minimize **coherence**:

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$$\kappa := \max_{\substack{i,j\in\{1,\dots,N\}\\ i\neq j}} |\langle\varphi_i,\varphi_j\rangle|.$$

Geometrically, this problem amounts to packing lines through the origin. Given their resemblance to error-correcting codes, it comes as no surprise that ensembles of minimal coherence, called **Grassmannian frames**, lead to communication protocols with minimal cross-talk. Since their introduction by Strohmer and Heath in 2003 [14], Grassmannian frames have received considerable attention in the finite frame theory community [7].

There were several precursors to the modern study of Grassmannian frames. One notable example is a 1974 paper of Welch [18], which provides various lower bounds for coherence in terms of the dimension M of $\mathcal{H}$. This first of these bounds is

$$\kappa \geq \sqrt{\frac{N-M}{M(N-1)}}, \tag{1.1}$$

cf. Rankin's 1956 paper [13]. This has since been dubbed the **Welch bound**, and one can show that the coherence of an ensemble of unit-length vectors meets equality in the Welch bound precisely when the ensemble forms an **equiangular tight frame** [14], meaning $|\langle\varphi_i,\varphi_j\rangle|$ is some fixed constant for every choice of i and j with $i\neq j$ (i.e., the vectors are "equiangular") and furthermore, the operator $\mathbf{S}\colon\mathcal{H}\to\mathcal{H}$ defined by

$$\mathbf{S}x := \sum_{i=1}^{N}\langle x,\varphi_i\rangle\varphi_i \qquad \forall x\in\mathcal{H}$$

is some multiple of the identity operator (i.e., the ensemble forms a so-called "tight frame"). By exhibiting equality in the Welch bound, equiangular tight frames (ETFs) are necessarily Grassmannian.

The significance of equiangularity brings us to another precursor of sorts. In 1966, van Lint and Seidel [16] first introduced a useful identification between real equiangular ensembles and graphs, in which vertices correspond to vectors, and edges are drawn according to the sign of the corresponding inner product. Since the Gram matrix $\mathbf{G} = [\langle\varphi_i,\varphi_j\rangle]_{i,j=1}^{N}$ can be expressed in terms of the adjacency matrix $\mathbf{A}$ of this graph, one may then identify spectral properties of $\mathbf{A}$ with those of $\mathbf{G}$. This is particularly important in the case of real ETFs, as the tightness criterion implies that $\mathbf{G}^2$ is a multiple of $\mathbf{G}$, which in turn forces $\mathbf{A}$ to satisfy a related quadratic equation. Indeed, more recent treatments of this identification [17,9] have established a one-to-one correspondence between real ETFs and a family of **strongly regular graphs** (SRGs), namely, graphs with the property that every vertex has the same number k of neighbors, that adjacent vertices have the same number λ of common neighbors, and that non-adjacent vertices also have the same number μ of common neighbors. Given the maturity of the SRG literature, this identification has compelled finite frame theorists to direct their attention toward complex ETFs, where open problems abound.