Journal of Combinatorial Theory, Series A 153 (2018) 98–121

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

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Prolific permutations and permuted packings: Downsets containing many large patterns

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A R T I C L E   I N F O

Article history:
Received 29 November 2016

Keywords:
Permutation
Pattern
Pattern poset
Downset
Prolific permutation
Packing
Permuted packing

A B S T R A C T

A permutation of n letters is k-prolific if each $(n-k)$-subset of the letters in its one-line notation forms a unique pattern. We present a complete characterization of k-prolific permutations for each k, proving that k-prolific permutations of m letters exist for every $m \geqslant k^2/2 + 2k + 1$, and that none exist of smaller size. Key to these results is a natural bijection between k-prolific permutations and certain "permuted" packings of diamonds.

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1. Introduction

The set of permutations of $[n] = \{1, 2, \ldots, n\}$ is denoted S_n. We write a permutation $\sigma \in S_n$ as a word over $[n]$ in one-line notation, $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$, and say that such

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[1] Research partially supported by a Simons Foundation Collaboration Grant for Mathematicians, number 277603, and by a DePaul University Faculty Summer Research Grant.

http://dx.doi.org/10.1016/j.jcta.2017.08.006
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a permutation σ has *size* n. If π_1 and π_2 are words of the same size over $\mathbb{R}$, then we write $\pi_1 \approx \pi_2$ to denote that their letters appear in the same relative order. This prompts the classical notion of pattern containment.

Definition 1.1. Consider $\pi \in S_r$. A permutation $\sigma \in S_n$ *contains* the *pattern* π if there are indices $1 \leqslant i_1 < \cdots < i_r \leqslant n$ such that $\sigma(i_1)\cdots\sigma(i_r) \approx \pi$. If σ contains π, we write $\pi \preceq \sigma$. If σ does not contain π, then σ *avoids* π.

From this, it is natural to define the "pattern poset" on permutations.

Definition 1.2. Let the *pattern poset*, $\mathcal{P}$, be the graded poset over $\bigcup_{k\geqslant 1} S_k$, ordered by the containment relation $\preceq$.

By definition, the elements of rank k in $\mathcal{P}$ are exactly the elements of S_k.

This paper is concerned with principal downsets of this poset, that is, with the sets of patterns which lie below a given permutation. In particular, we examine those permutations whose downset is as large as possible in the upper ranks.

This is related to problems of *pattern packing* [1,14], which seek to maximize the total number of distinct patterns contained in a permutation, and to problems of *superpatterns* [6,8,9,14], which are concerned with determining the size of the smallest permutations whose downset contains every permutation of some fixed size. Other related work addresses permutation *reconstruction* [7,15,16], establishing when permutations are uniquely determined by the (multi)set of large patterns they contain. The reader is referred to the books by Bóna [3] and Kitaev [13] for an overview of problems related to the permutation pattern poset.

It follows immediately from the definition of $\mathcal{P}$ that, for a permutation $\sigma \in S_n$, there are at most $\binom{n}{k}$ distinct permutations $\pi \preceq \sigma$ that lie exactly k ranks below σ in $\mathcal{P}$, since each such permutation is obtained from σ by the deletion of k letters from the one-line notation for σ. Our interest is in those permutations of size n which contain maximally many patterns of size $n-k$.

Definition 1.3. Fix positive integers $n > k \geqslant 1$. A permutation $\sigma \in S_n$ is *k-prolific* if

$$\left|\left\{\pi \in S_{n-k} \,:\, \pi \preceq \sigma\right\}\right| = \binom{n}{k}.$$

Clearly, not every permutation is k-prolific. As a trivial example, the identity permutation $12\cdots n \in S_n$ contains only one pattern of each size, and thus is never k-prolific for any $k < n$.

Prolific permutations were previously investigated by the second author in [10]. The present work corrects and significantly improves upon the results presented there.

It is helpful to consider permutations from a graphical perspective.

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