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Coloring graphs with forbidden minors

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ABSTRACT

Hadwiger's conjecture from 1943 states that for every integer $t \geq 1$, every graph either can be t-colored or has a subgraph that can be contracted to the complete graph on $t+1$ vertices. As pointed out by Paul Seymour in his recent survey on Hadwiger's conjecture, proving that graphs with no K_7 minor are 6-colorable is the first case of Hadwiger's conjecture that is still open. It is not known yet whether graphs with no K_7 minor are 7-colorable. Using a Kempe-chain argument along with the fact that an induced path on three vertices is dominating in a graph with independence number two, we first give a very short and computer-free proof of a recent result of Albar and Gonçalves and generalize it to the next step by showing that every graph with no K_t minor is $(2t-6)$-colorable, where $t \in \{7, 8, 9\}$. We then prove that graphs with no K_8^- minor are 9-colorable, and graphs with no $K_8^=$ minor are 8-colorable. Finally we prove that if Mader's bound for the extremal function for K_t minors is true, then every graph with no K_t minor is $(2t-6)$-colorable for all $t \geq 6$. This implies our first result. We believe that the Kempe-chain method we have developed in this paper is of independent interest.

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1. Introduction

All graphs in this paper are finite and simple. A graph H is a *minor* of a graph G if H can be obtained from a subgraph of G by contracting edges. We write $G > H$ if H is a minor of G. In those circumstances we also say that G has an H *minor*.

Our work is motivated by the following Hadwiger's conjecture [6], which is perhaps the most famous conjecture in graph theory, as pointed out by Paul Seymour in his recent survey [18].

Conjecture 1.1. *For every integer $t \geq 1$, every graph with no K_{t+1} minor is t-colorable.*

Hadwiger's conjecture is trivially true for $t \leq 2$, and reasonably easy for $t = 3$, as shown by Dirac [3]. However, for $t \geq 4$, Hadwiger's conjecture implies the Four Color Theorem. Wagner [22] proved that the case $t = 4$ of Hadwiger's conjecture is, in fact, equivalent to the Four Color Theorem, and the same was shown for $t = 5$ by Robertson, Seymour and Thomas [16]. Hadwiger's conjecture remains open for $t \geq 6$. As pointed out by Paul Seymour [18] in his recent survey on Hadwiger's conjecture, proving that graphs with no K_7 minor are 6-colorable is thus the first case of Hadwiger's conjecture that is still open. It is not even known yet whether every graph with no K_7 minor is 7-colorable. Kawarabayashi and Toft [11] proved that every graph with no K_7 or $K_{4,4}$ minor is 6-colorable. Jakobsen [8,9] proved that every graph with no $K_7^=$ minor is 6-colorable and every graph with no K_7^- minor is 7-colorable, where for any integer $p > 0$, K_p^- denotes the graph obtained from K_p by removing one edge, and $K_p^=$ denotes the family of two non-isomorphic graphs each obtained from K_p by removing two edges. Note that a graph has no $K_p^=$ minor if it does not contain any of the two graphs in $K_p^=$ as a minor; and a graph G has a $K_p^=$ minor or $G > K_p^=$ if G contains one of the graphs in $K_p^=$ as a minor. For more information on Hadwiger's conjecture, the readers are referred to an earlier survey by Toft [21] and a very recent informative survey due to Seymour [18].

Albar and Gonçalves [1] recently proved the following:

Theorem 1.2. (Albar and Gonçalves [1]) *Every graph with no K_7 minor is 8-colorable, and every graph with no K_8 minor is 10-colorable.*

The proof of Theorem 1.2 is computer-assisted and not simple. In this paper, we apply a Kempe-chain argument (see Lemma 1.7 below), along with the fact that an induced path on three vertices is dominating in a graph with independence number two, to give a much shorter and computer-free proof of Theorem 1.2. In addition, we generalize it to the next step by proving the following.

Theorem 1.3. *Every graph with no K_t minor is $(2t-6)$-colorable, where $t \in \{7, 8, 9\}$.*

We want to point out that our proof of Theorem 1.3 does not rely on Mader's deep result on the connectivity of contraction-critical graphs (see Theorem 1.8 below).

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