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Monochromatic cycle partitions of graphs with large minimum degree

Louis DeBiasio[1], Luke L. Nelsen

Department of Mathematics, Miami University, Oxford, OH 45056, United States

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ABSTRACT

Lehel conjectured that in every 2-coloring of the edges of K_n, there is a vertex disjoint red and blue cycle which span $V(K_n)$. Łuczak, Rödl, and Szemerédi proved Lehel's conjecture for large n, Allen gave a different proof for large n, and finally Bessy and Thomassé gave a proof for all n.
Balogh, Barát, Gerbner, Gyárfás, and Sárközy proposed a significant strengthening of Lehel's conjecture where K_n is replaced by any graph G with $\delta(G) > 3n/4$; if true, this minimum degree condition is essentially best possible. We prove that their conjecture holds when $\delta(G) > (3/4 + o(1))n$. Our proof uses Szemerédi's regularity lemma along with the absorbing method of Rödl, Ruciński, and Szemerédi by first showing that the graph can be covered with monochromatic subgraphs having certain robust expansion properties.

1. Introduction

For the purposes of this paper, we consider the empty set, a single vertex, and an edge as cycles on 0, 1, and 2 vertices respectively. By an r-coloring of a graph G, we mean a partition of its edge set into at most r parts (i.e. exactly r parts, some of which may

E-mail addresses: debiasld@miamioh.edu (L. DeBiasio), luke.nelsen@ucdenver.edu (L.L. Nelsen).
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be empty). Given an r-colored graph G, a partition of G into monochromatic cycles is a collection of vertex disjoint monochromatic cycles which together span $V(G)$. We denote a path or cycle on k vertices by P^k and C^k respectively (subscripts will be reserved for colors).

In 1967, Gerencsér and Gyárfás [11] exactly determined the Ramsey number for all pairs of paths. In the symmetric case (when the paths have the same length), the result can be stated as follows.

Theorem 1.1 *(Gerencsér, Gyárfás). Every 2-coloring of K_n contains a monochromatic P^k with $k > 2n/3$.*

In 1973, Rosta [26] and independently, Faudree and Schelp [9] exactly determined the Ramsey number for all pairs of cycles, which gave an analog of Theorem 1.1 for cycles. Later, this was slightly refined by Faudree, Lesniak, and Schiermeyer [10] to give the following best possible result about long monochromatic cycles.

Theorem 1.2 *(Faudree, Lesniak, Schiermeyer). For $n \geq 6$, every 2-coloring of K_n contains a monochromatic C^k with $k \geq 2n/3$.*

In [11], Gerencsér and Gyárfás wrote a small, but historically influential, footnote which contained the seed of a new "Ramsey-type" partitioning problem. In the footnote was a simple proof that every 2-coloring of K_n has a cycle on n vertices which is the union of a blue path and a red path (which in turn contains a monochromatic $P^{\lceil n/2 \rceil}$). In a 2-colored K_n, a cycle on n vertices which is the union of a blue path and a red path immediately gives a partition of K_n into two monochromatic paths; from this one can easily deduce that K_n has a partition into a vertex disjoint monochromatic cycle and path of different colors. Later, Lehel (see [2] and [8]) conjectured that every 2-coloring of K_n has a partition into a red cycle and blue cycle (note the requirement that the cycles have different colors).

Lending further support to Lehel's conjecture, Gyárfás [12] proved that in every 2-coloring of K_n there is a red cycle and a blue cycle which span the vertex set and have at most one common vertex. Łuczak, Rödl, and Szemerédi [23] proved Lehel's conjecture for large n and later Allen [1] gave a different proof of Lehel's conjecture for smaller, but still large n. Finally, Bessy and Thomassé [5] proved Lehel's conjecture for all n.

Theorem 1.3 *(Bessy, Thomassé). Every 2-coloring of K_n has a partition into a red cycle and blue cycle.*

Schelp [27] raised the general problem of determining whether results such as Theorem 1.1, Theorem 1.2, and Theorem 1.3, which are about complete graphs, actually hold for graphs with sufficiently large minimum degree. In particular he conjectured that the conclusion of Theorem 1.1 still holds if K_n is replaced by any graph G with