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Luzin's topological problem

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ABSTRACT

A resolvably measurable function is a real-valued function for which the preimage of each open set is resolvable. It is shown that resolvably measurable functions $f : X \subset \mathbf{R} \to Y \subset \mathbf{R}$ (a subclass of Δ_2^0-measurable functions) have a decomposition into countably many continuous restrictions.

We study a class of particular functions $f : X \to Y$ of the first Baire class and their decomposition to countably many continuous restrictions. Luzin formulated the problem whether Borel measurable functions have such a decomposition. It is known that this is not the case even for functions of the first Baire class [1]. Jayne and Rogers showed in [6] that such a decomposition exists, for sufficiently regular (e.g., analytic) space X, for a smaller class of Δ_2^0-measurable functions f (which coincides with the class of resolvable–measurable functions in the case of complete metric spaces X); moreover, the restrictions may be restrictions to closed subsets of X in such a case. Further deep improvements can be found in a paper by S. Solecki [15] (for more information see papers [7] by M. Kačena, L. Motto Ros, B. Semmes, [2] by T. Banakh and B. Bokalo, [14] by J. Pawlikowski and M. Sabok, and very recenly the works by S. Medvedev).

Note that our interest in resolvably measurable functions arose after the remarkable results of S. Gao and V. Kieftenbeld, P. Holický and R. Pol, who proved that open–resolvable functions preserve Polish spaces [3], [4], [5]. Since a function f is open–resolvable if and only if its inverse f^{-1} is (multivalued) resolvably

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measurable, the question of the decomposition of an open–resolvable function into open [13] is similar to the question of the decomposition of a resolvably measurable function into continuous ones.

In the statement (in French) of Luzin's problem from the article by L. Keldiš [8], presented for publication by Luzin himself, there are no constraints on the domain of definition X of the function f, and many authors consider X as a topological, not necessarily analytic, space. In addition, taking into account that the resolvable space is a topological rather than descriptive concept, we will speak in such situations of Luzin's topological problem, a solution to which is given by Theorem 1.

Recall that a subset E of a space X is *resolvable* if for each nonempty subset F, closed in X, there holds:

$$\overline{F \cap E} \cap \overline{F \setminus E} \neq F.$$

Resolvable sets are known as all sets in the difference hierarchy [9, §12,II]. In each Polish space, resolvable sets are exactly the sets that are both F_σ and G_δ (a brief account can be found in [3]).

We say that a function f is *resolvably measurable* if the inverse images of open sets are resolvable sets.

Recall that a function $f : X \to Y$ is *countably continuous* if X admits a countable cover $\mathcal{C}$ of X such that for each $C \in \mathcal{C}$ the restriction $f|C$ is continuous [10].

Theorem 1. *Each resolvably measurable function $f : X \to Y$ between subsets X and Y of the Cantor set $\mathbf{C}$ is countably continuous.*

Theorem 1 continues the series of publications [11], [12], [13] by the author.

Corollary 1. *Each resolvably measurable function $f : X \to Y$ between subsets X and Y of real numbers $\mathbf{R}$ is countably continuous.*

Indeed, let us consider two countable sets: $Q_1 = f(X \cap \mathbf{Q})$ and $Q_2 = f(X) \cap \mathbf{Q}$, where $\mathbf{Q}$ is the set of rational numbers and denote $Y_1 = Y \setminus (Q_1 \cup Q_2)$, $X_1 = f^{-1}(Y_1)$. The restriction $f|X_1$ is a resolvably measurable function between two subspaces of the space of irrational numbers, which is homeomorphic to a subspace of $\mathbf{C}$. By Theorem 1, $f|X_1$ is countably continuous and, since $Y \setminus Y_1$ is countable, f is countably continuous. $\square$

In the rest of this paper it will be assumed that X and Y are subsets of the Cantor set $\mathbf{C}$.

1. Definition of X^*

Let $X^* = X \setminus \bigcup_{W \in \mathcal{B}} R(W)$, where $\mathcal{B}$ is a countable clopen base of $\mathbf{C}$ and

$$R(W) = \bigcup_{U \in \mathcal{B}} \{U \cap f^{-1}(W) : f|U \cap f^{-1}(W) \text{ is countably continuous}\}.$$

Lemma 1.1. *Let $f : X \to Y$ be a function that is not countably continuous*; *then, for every nonempty set $I_1 = U \cap f^{-1}(W) \cap X^*$, where W and $U \in \mathcal{B}$, the restriction $f|I_1$ is not countably continuous.*

Proof. Since the base $\mathcal{B}$ is countable, f is countably continuous on $X \setminus X^* = \bigcup_{W \in \mathcal{B}} R(W)$ and, thus, on its subset $I_2 = f^{-1}(W) \setminus X^*$. If we suppose the opposite to the assertion of Lemma 1.1, namely, that $f|I_1$ is countably continuous, then f is countably continuous on $I_1 \cup I_2 = (U \cap f^{-1}(W) \cap X^*) \cup (f^{-1}(W) \setminus X^*)$, and thus, on its subset $(U \cap f^{-1}(W) \cap X^*) \cup ((U \cap f^{-1}(W)) \setminus X^*) = U \cap f^{-1}(W)$. Since $U \cap f^{-1}(W) \subset R(W)$, we obtain:

$$U \cap f^{-1}(W) \cap X^* \subset R(W) \cap X^*.$$