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A direct characterization of the monotone convergence space completion

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ABSTRACT

For a T_0-space X and the set $\Gamma(X)$ of all closed subsets of X, the d-closure of all point closures of X in $\Gamma(X)$ endowed with the lower Vietoris topology is called the standard D-completion of X. In this paper, by introducing the notion of a tapered set, we present a direct characterization of the D-completion: the set of all tapered closed subsets of X endowed with the lower Vietoris topology is exactly the standard D-completion of X.

1. Introduction

The notion of a directed set plays an important role in modeling of computation. It is known that a T_0-space X is called sober if every irreducible closed subset is the closure of a singleton set. Analogously, if the closure of each directed subset of X (in the specialization order) is a point closure, then X is called a monotone convergence space. The concept of a monotone convergence space was introduced by Wyler [4] under the name of a d-space. Wyler also gave a construction for the d-space completion of every T_0-space X, later called the D-completion: the smallest sub-dcpo of $\Gamma(X)$ (the set of all closed subsets of X) which contains all point closures, together with the lower Vietoris topology is a D-completion of X. This completion is a category reflection of the category of T_0-spaces onto the full subcategory of d-spaces. Ershov [1] later showed that for any subspace X_0 of a d-space X, the smallest d-subspace of X containing X_0 is a D-completion of X_0. In [3], Keimel and Lawson gave a general categorical construction via reflection functors for various completions of T_0-spaces, and emphatically investigated the D-completion as a special case.

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It is known that the space of all irreducible closed subsets of a T_0-space X is a sobrification of X. One question arises naturally: can we give a direct characterization of the D-completion? That is to say, what type of closed subsets of X is exactly the smallest sub-dcpo of $\Gamma(X)$ containing all point closures? This paper is set to answer this question.

2. Preliminaries

This section is an introduction to the concepts and results about monotone convergence spaces and the D-completion. For more details, refer to [2,3]. The topological spaces considered in this paper are all T_0-spaces, and each space is endowed with the specialization order.

Definition 2.1. A T_0-space is called a monotone convergence space (or D-space) if every directed subset has a supremum to which it converges.

Equivalently, a T_0-space is a monotone convergence space iff it is a dcpo and all open sets are Scott open.

For a topological space X, we denote the set of all open sets of X by $\mathcal{O}(X)$ and that of closed sets by $\Gamma(X)$. And for any subset A of X, $cl(A)$ denotes the closure of A. Notice that a point closure $cl(\{x\}) = \downarrow x$. Let $\Gamma(X)$ be endowed with the lower Vietoris topology $\mathcal{O}(\Gamma(X))$, generated by subbasic open sets of the form

$$\Diamond U := \{A \in \Gamma(X) : A \bigcap U \neq \emptyset\},$$

where U ranges over $\mathcal{O}(X)$. A standard argument shows that the specialization order on $\Gamma(X)$ is the inclusion relation and $\Gamma(X)$ is a monotone convergence space. The subspace

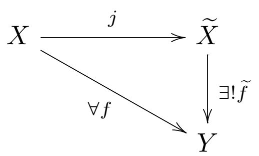
$$X^s := \{A \in \Gamma(X) : A \text{ is irreducible}\}$$

of $\Gamma(X)$, together with the function $\eta_X^s : X \to X^s$ defined by $\eta_X^s(x) = cl(\{x\})$, is *the standard sobrification* of the space X.

The d-closure of a subset A of a monotone convergence space X, denoted by $cl_d(A)$, is the smallest sub-dcpo containing A. The set A is called a d-dense subset of X if X is the smallest sub-dcpo containing A, i.e., $cl_d(A) = X$.

Definition 2.2. A monotone convergence space $\widetilde{X}$ together with a topological embedding $j : X \to \widetilde{X}$ with $j(X)$ a d-dense subset of $\widetilde{X}$ is called a D-completion of the space X.

A D-completion $(\widetilde{X}, j)$ of X satisfies the universal property: for every continuous function $f : X \to Y$ mapping into a monotone convergence space Y, there exists a unique continuous function $\widetilde{f} : \widetilde{X} \to Y$ such that $f = \widetilde{f} \circ j$, i.e., the following diagram commutes:



Define $\Psi(X) := \{cl(\{x\}) : x \in X\}$. Let $cl_d(\Psi(X))$ be the d-closure of $\Psi(X)$ in X^s, endowed with the subspace topology of X^s, and η_X be the corestriction of η_X^s from X to $cl_d(\Psi(X))$. Then $(cl_d(\Psi(X)), \eta_X)$ is called *the standard D-completion* of X. Since X^s is a sub-dcpo and subspace of $\Gamma(X)$, we have that $cl_d(\Psi(X))$ is also the d-closure of $\Psi(X)$ in $\Gamma(X)$ and a subspace of $\Gamma(X)$.