Topology and its Applications 230 (2017) 131–138

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Virtual Special Issue – Proceedings of the First Pan-Pacific International Conference on Topology and Applications (PPICTA)

A random link via bridge position is hyperbolic

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A R T I C L E   I N F O

Article history:
Received 24 May 2016
Received in revised form 11 August 2016
Accepted 11 August 2016
Available online 26 August 2017

MSC:
57M25
20F36
60G50

Keywords:
Random link
Random walk
Bridge presentation
Hyperbolic

A B S T R A C T

We show that a random link defined by random bridge splitting is hyperbolic with asymptotic probability 1.

1. Introduction

In recent low-dimensional topology, to study 3-manifolds and knot via *random methods* could be a hot topic now.

As a pioneering work, in [1], Dunfield and Thurston introduced a model of random 3-manifolds by using random walks on the mapping class group of a surface, and a theory of random 3-manifolds has started. Actually they considered random Heegaard splittings by gluing a pair of handlebodies by the result of a random walk in the mapping class group.

Later random Heegaard splittings are studied extensively by Maher in [2]. In particular, he showed that a 3-manifold obtained by a random Heegaard splitting is hyperbolic with asymptotic probability 1.

As a natural generalization, a random bridge decomposition for a link in the 3-sphere S^3 was considered and studied by the second author in [3]. There he computed the expected value of the number of components

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http://dx.doi.org/10.1016/j.topol.2017.08.028

of a random link, via the random braid model and the random bridge presentation model. A further study to this for random braid model is given in [4].

Moreover, for random braid model, Ma showed in [5] that, for a random walk on the braid group, the probability that the link appearing as the braid closure is hyperbolic converges to 1.

In this paper, as a generalization to the result of Maher, and a dual result of Ma, we show that a link obtained by a random bridge decomposition is hyperbolic with asymptotic probability 1.

To state our result precisely, we set up our terminology. Let $B = B_+$ be a 3-ball with trivial (i.e., boundary-parallel) n-strings τ_+ in it, and B_- and τ_- their mirror images. Regarding (B_*, τ_*) as the trivial tangle, the boundary of (B_*, τ_*), meaning that $\partial B_* - \tau_*$, is a $2n$-punctured sphere $S_{0,2n}$ for $* = +$ or $-$. Let us denote by $\mathfrak{M}_{0,2n}$ the (orientation-preserving) mapping class group of $S_{0,2n}$. Then, for $\varphi \in \mathfrak{M}_{0,2n}$ with a representative $f : S_{0,2n} \to S_{0,2n}$, $(B_+ - \tau_+) \cup_f (B - -\tau_-)$ gives the complement of a link L in the 3-sphere S^3 so that the sets of strings τ_+ and τ_- are glued together to get L. We denote by $L = \bar{\varphi}$, and call it a link with an n-bridge presentation $(B_+, \tau_+) \cup_\varphi (B_-, \tau_-)$.

Now the statement of our main result is the following.

Theorem 1.1. *Let w_k be a random walk in the mapping class group $\mathfrak{M}_{0,2n}$ of $2n$-punctured sphere for $n \geq 3$. Suppose that w_k is generated by a finitely supported probability distribution μ on $\mathfrak{M}_{0,2n}$, whose support generates a semi-group containing a complete subgroup. Then the probability that the link $\overline{w_k}$ is hyperbolic tends to 1 as $k \to \infty$.*

Here a subgroup of the mapping class group is called *complete* if the endpoints of its pseudo-Anosov elements are dense in the space of projective measured foliations on the surface. We here omit the details about the mapping class groups on surfaces and the random walks on groups. Please refer to [2, Section 2].

We remark that we except the special case $n = 2$, that is, the 2-bridge link case. For technical reason, it should be treated separately, but one could obtain the same result in the same way.

Our proof is similar to the approach of Maher [2], but in the approach, we show two results on the Hilden groups, which are similar to results in the handlebody groups, which should be interesting and deserve to write down for the purpose of knot theory.

2. Outline of proof

In this section, we give an outline of the proof of Theorem 1.1. Our proof just runs along the same line as the proof of [2, Theorem 1.1] by Maher, which presents the same result for closed 3-manifolds.

As in Theorem 1.1, let w_k be a random walk in the mapping class group $\mathfrak{M}_{0,2n}$ of $2n$-punctured sphere for $n \geq 3$. We suppose that w_k is generated by a finitely supported probability distribution μ on $\mathfrak{M}_{0,2n}$, whose support generates a semi-group containing a complete subgroup.

We primarily want to apply the following result due to Maher to our setting.

> [2, Theorem 5.5] Let G be the mapping class group of a orientable surface. Consider a random walk generated by a finitely supported probability distribution μ on G, and whose support generates a semi-group containing a complete subgroup of G. Let X be a quasi-convex subset of the relative space $\hat{G}$, whose limit set has measure zero with respect to both harmonic measure and reflected harmonic measure. Then there is a constant $\ell > 0$ such that
>
> $$\mathbb{P}\left(\left|\frac{1}{n}\hat{d}(X, w_n X) - \ell\right| \leq \epsilon\right) \to 1 \text{ as } n \to \infty,$$
>
> for all $\epsilon > 0$, where $\mathbb{P}$ is the probability measure determined by the probability distribution μ and $\hat{d}(X, w_n X)$ is the minimum distance between X and $w_n X$.