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Converting virtual link diagrams to normal ones ☆

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A B S T R A C T

A virtual link diagram is called normal if the associated abstract link diagram is checkerboard colorable, and a virtual link is normal if it has a normal diagram as a representative. In this paper, we introduce a method of converting a virtual link diagram to a normal virtual link diagram by use of the double covering technique. We show that the normal virtual link diagrams obtained by this method from two equivalent virtual link diagrams are related by generalized Reidemeister moves and Kauffman flypes.

1. Introduction

L. H. Kauffman [8] introduced virtual knot theory, which is a generalization of knot theory based on Gauss diagrams and link diagrams in closed oriented surfaces. Virtual links correspond to stable equivalence classes of links in thickened surfaces [2,5]. A virtual link diagram is called normal if the associated abstract link diagram is checkerboard colorable (§ 2). A virtual link is called normal if it has a normal diagram as a representative. Every classical link diagram is normal, and hence the set of classical link diagrams is a subset of that of normal virtual link diagrams. The set of normal virtual link diagrams is a subset of that of virtual link diagrams. The f-polynomial (Jones polynomial) is an invariant of a virtual link [8]. It is shown in [3] that the f-polynomial of a normal virtual link has a property that the f-polynomial of a classical link has. This property may make it easier to define Khovanov homology of virtual links as stated in O. Viro [10].

In this paper, we introduce a method of converting a virtual link diagram to a normal virtual link diagram by use of the double covering technique defined in [6]. We show that the normal virtual link diagrams obtained from two equivalent virtual link diagrams by our method are related by generalized Reidemeister moves and Kauffman flypes. This result is shown in a more general setting using cut systems (Theorem 7). We also discuss cut systems on virtual link diagrams and answer a question given by H. Dye.

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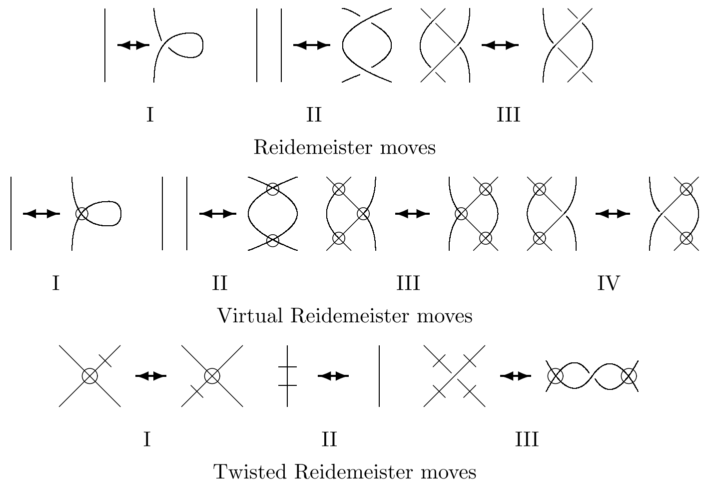

Fig. 1. Generalized Reidemeister moves and twisted Reidemeister moves.

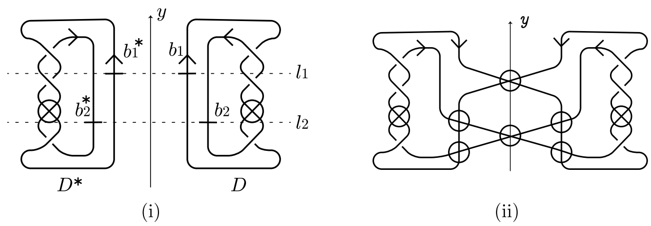

Fig. 2. The double covering of a twisted link diagram.

In Section 4, the odd writhe of a virtual knot is interpreted in terms of the linking number of the converted link diagram.

2. Definitions and main results

A *virtual link diagram* is a generically immersed, closed and oriented 1-manifold in $\mathbb{R}^2$ with information of positive, negative or virtual crossing, on its double points. A *virtual crossing* is an encircled double point without over-under information [8]. A *twisted link diagram* is a virtual link diagram, possibly with *bars* on arcs [1]. A *virtual link* (or *twisted link*) is an equivalence class of virtual (or twisted) link diagrams under Reidemeister moves and virtual Reidemeister moves (or Reidemeister moves, virtual Reidemeister moves and twisted Reidemeister moves) depicted in Figs. 1. We call Reidemeister moves and virtual Reidemeister moves *generalized Reidemeister moves.*

We recall from [6] the double covering diagram of a twisted link diagram. Let D be a twisted link diagram. Assume that D is on the right of the y-axis in the xy-plane and all bars are parallel to the x-axis with disjoint y-coordinates. Let D^* be the twisted link diagram obtained from D by reflection with respect to the y-axis and switching the over-under information of all classical crossings of D. Let $B = \{b_1, \dots, b_k\}$ be the set of bars of D and for $i \in \{1, \dots, k\}$, we denote by b_i^* the bar of D^* corresponding to b_i. See Fig. 2 (i). For horizontal lines $l_1, \dots, l_k$ such that l_i contains b_i and the corresponding bar b_i^* of D^*, we replace each part of $D \amalg D^*$ in a neighborhood of $N(l_i)$ for each $i \in \{1, \dots, k\}$ as in Fig. 3. We denote by $\psi(D)$ the virtual link diagram obtained this way.

For example, for the twisted link diagram D depicted as in Fig. 2 (i), the virtual link diagram $\psi(D)$ is as in Fig. 2 (ii). We call this diagram $\psi(D)$ the *double covering diagram* of D. Then we have the followings.