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Small connected neighborhoods containing the diagonal of a product

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ABSTRACT

We characterize chainable continua X for which the diagonal Δ in $X \times X$ has small connected neighborhoods. That means that for each open subset W of $X \times X$ with $\Delta \subset W$, there exists a connected subset M of $X \times X$ such that $\Delta \subset \mathrm{int}_{X \times X}(M) \subset M \subset W$.

1. Introduction

A *continuum* is a compact connected metric space with more than one point.

Let X be a continuum. A *subcontinuum* of X is a nonemtpy closed connected subspace of X, so the one-point sets of X are subcontinua of X. The *hyperspace of subcontinua* of X is denoted by $C(X)$ and is considered with the Hausdorff metric H [5, Definition 2.1].

Denote by Δ_X the diagonal of $X \times X$. That is,

$$\Delta_X = \{(x, x) \in X \times X : x \in X\}.$$

The continuum X has the property that *the diagonal of $X \times X$ has small connected open neighborhoods (diascoopen)* provided that for each open subset W of $X \times X$ such that $\Delta_X \subset W$, there exists an open connected subset U of $X \times X$ such that $\Delta_X \subset U \subset W$. We say that X has the property that *the diagonal*

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of $X \times X$ has small connected neighborhoods (diascone) provided that for each open subset W of $X \times X$ such that $\Delta_X \subset W$, there exists a connected subset M of $X \times X$ such that $\Delta_X \subset \text{int}_{X\times X}(M) \subset M \subset W$.

Given a family of metric continua $\{X_\alpha : \alpha \in J\}$, the product $X = \prod_{\alpha\in J} X_\alpha$ has the property of *full projection implies arbitrary small connected open neighborhoods (fupcon)* provided that for every subcontinuum M and open subset W of X such that $M \subset W$ and $\pi_\alpha(M) = X_\alpha$ for each $\alpha \in J$ (π_α is the α^{th}-projection), we have that there exists an open connected subset U of X such that $M \subset U \subset W$.

Clearly, each product of locally connected continua has the fupcon property. It is also clear that if X is a continuum and $X \times X$ has the fupcon property, then X has the diascone property. The converse is not true even for indecomposable chainable continua (see comments at the end of the paper). It is also possible to define the property fupcon with neighborhoods instead of open neighborhoods. However, in [4, Lemma 1] it was proved that both definitions are equivalent.

It is easy to prove that if X has the diascone property, then $C(X \times X)$ is connected im kleinen at Δ_X.

As we can see in the examples in [2] and [4], and in the examples of this paper there many continua which are very far from being locally connected but they have the diascone property.

Obviously, if a continuum has the diascoopen property, then it has the diascone property. We do not know if the converse holds.

Products with the fupcon property have been studied in [2] and [4].

It is known that:

A. any product of Knaster continua has the fupcon property [2, Theorem 4.1],
B. any product of pseudo-arcs has the fupcon property [2, Theorem 4.4],
C. the product of a pseudo-arc with any product of Knaster continua has the fupcon property [4, Corollary 9],
D. there exist indecomposable chainable continua whose product does not have the fupcon property [4, Example 11], and
E. a product of a solenoid with itself does not have the diascone property [2, Corollary 4.3],
F. a product of a Knaster continuum and a solenoid has the fupcon property (proved recently by A. Illanes, J. M. Martínez-Montejano and K. Villarreal, in a paper under preparation).

It is not known if the following product have the fupcon property:

G. the product of two non-homeomorphic solenoid groups [2, p. 230].

A *chain* in a continuum X is a finite family $\{U_1, \ldots, U_n\}$ of open subsets of X such that $U_i \cap U_j \neq \emptyset$ if and only if $|i - j| \leq 1$. The continuum X is *chainable* provided that for each $\varepsilon > 0$, there exists a chain $\{U_1, \ldots, U_n\}$ such that $X = U_1 \cup \ldots \cup U_n$ and diameter$(U_i) < \varepsilon$ for each $i \in \{1, \ldots, n\}$.

A proper subcontinuum K of a continuum X is an R^2-*continuum* [3, Definition 1.2, p. 75] provided that there exists an open subset U of X and sequences of components $\{A_n\}_{n=1}^\infty$ and $\{B_n\}_{n=1}^\infty$ of U such that $K \subset U$, $\{\text{cl}_X(A_n)\}_{n=1}^\infty$ and $\{\text{cl}(B_n)\}_{n=1}^\infty$ converge (in the hyperspace $C(X)$) to respective subcontinua A and B of X and $K = A \cap B$. Notice that $K \neq X$ implies that $U \neq X$.

In [1, Conjecture 8, p. 260], D. P. Bellamy conjectured that each chainable continuum has the diascoopen property. In this paper we prove that Bellamy's conjecture does not hold and we characterize chainable continua having the diascone property.

We prove that a chainable continuum X has the diascone property if and only if X contains no R^2-continua.