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On well-filtered spaces and ordered sets

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ABSTRACT

A topological space is well-filtered if any filtered family of compact sets with intersection in an open set must have some member of the family contained in the open set. This well-known and important property automatically satisfied in Hausdorff spaces assumes a life of its own in the T_0-setting. Our main results focus on giving general sufficient conditions for a T_0-space to be well-filtered, particularly the important case of directed complete partially ordered sets equipped with the Scott topology.

1. Introduction

Driven by topologies arising in the spectral theory of rings and C^*-algebras, in the domain theory [2] of theoretical computer science, and in categorical topology, the theory of non-Hausdorff topological spaces has undergone substantial development, as the recent monograph by Goubault-Larrecq [1] documents. Spectral and domain theory are concerned with special classes of T_0-spaces, spaces in which any two points can be separated by an open set, and these will be the main concern in this paper. Many of the familiar concepts from general topology reappear in the T_0-setting, but frequently in an altered or nuanced manner. For example in the Hausdorff setting, compact subspaces satisfy the properties of being closed under finite intersections, being locally compact, and having nonempty intersections for filtered families of nonempty compact sets. The property of compactness alone turns out to be much weaker in the T_0-setting, and any of the three mentioned properties may fail for a compact set. Indeed the closest analog to a compact Hausdorff space in the T_0-setting is that of a stably compact space, a special type of compact space that also satisfies the other three conditions. One might say that topology in the T_0-setting can give deeper insight into the

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nature of topological properties by seeing what needs to be hypothesized to achieve these properties in this more general setting.

In this paper we uncover new insights into one of the extra properties needed to simulate in the T_0-setting the behavior of compact subsets in a Hausdorff space, namely the property of being well-filtered, which we define in the next section. The requirement that a space be Hausdorff, or more generally sober, are the only sufficient conditions that have been identified for a space to be well-filtered to the best of our knowledge. The main purpose of this paper is to identify other useful sufficient conditions that can be applied to existing and future examples and classes of examples, particularly examples outside the sphere of the intensively studied special class of dcpos consisting of continuous domains and quasidomains.

2. Monotone convergence spaces

Recall that a subset A of a topological space X is *saturated* if it is the intersection of all the open sets containing it. In a topological space X with at least T_1-separation every subset is saturated, so the notion is only interesting for those spaces that have singleton subsets that are not closed. A nonempty family of subsets of a set X is said to be *filtered* if any two members of the family contain some third member. A space is said to be *well-filtered* if for every filtered family $\mathcal{F}$ of compact saturated sets with intersection $\bigcap\mathcal{F}$ contained in some open set U, it follows that $F \subseteq U$ for some $F \in \mathcal{F}$ [2, Definition I-1.24.1], [1, Section 8.3.1]. This basic, well-known, and useful property of compact sets holds in any Hausdorff space and holds more generally for an important class of T_0-spaces, the sober spaces [2, Lemma II-1.19]. (Recall a T_0-space is sober if every closed set that is not the union of two smaller ones is the closure of a singleton set.) Conversely if X is locally compact, T_0, and well-filtered, then X is sober [2, Theorem II-1.21] [1, Proposition 8.3.8], but this is not true in general without local compactness.

The *order of specialization* for a T_0-space X is a partial order on X given by $x \le y$ if $x \in \overline{\{y\}}$. For $A \subseteq X$ we set $\uparrow A = \{y \in X : \exists x \in A, x \le y\}$; $\downarrow A$ is defined in an order dual fashion. A directed subset D of a partially ordered set P is a nonempty subset satisfying for every $d_1, d_2 \in D$, there exists $d_3 \in D$ such that $d_1, d_2 \le d_3$ and a directed subset of a T_0-space is one that is directed with respect to the order of specialization. A directed subset is said to converge to some point of a T_0-space if it converges in the sense of a net, where the directed set is viewed as a net indexed by itself. A T_0-space is called a *monotone convergence space* [2, Definition II-3.12] if every directed set has a supremum to which it converges.

Proposition 2.1. *A well-filtered T_0-space X is a monotone convergence space.*

Proof. Let D be a nonempty subset of X that is directed in the order of specialization with supremum e, and let U be an open set around e. Since D is directed, the family $\{\uparrow d : d \in D\}$ is filtered and one easily verifies that $\bigcap\{\uparrow d : d \in D\}$ consists of all upper bounds of D, which is contained in $\uparrow e$. Since X is well-filtered, $\uparrow d_0 \subseteq U$ for some $d_0 \in D$, and hence $\uparrow d \subseteq \uparrow d_0 \subseteq U$ for all $d \ge d_0$. □

A *coherent* topological space is one in which the intersection of two compact saturated sets is compact (the intersection is automatically saturated).

Proposition 2.2. *For a T_0-space X the following are equivalent.*

(1) *X is compact, coherent, and well filtered.*
(2) *X is compact with respect to the patch topology, the topology with a closed subbasis consisting of the closed sets in the given topology together with the compact saturated sets.*

Proof. (1)⇒(2): By the Alexandroff Subbasis Theorem, it suffices to show that every subbasic open cover $\mathcal{U}$ has a finite subcover. Let W be the union of all members of $\mathcal{U}$ that are open in X. If $W = X$, then