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Observations on some cardinality bounds ☆

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ABSTRACT

We try to find a common extension of two cardinal inequalities for Lindelöf spaces. Using an estimate of the number of G_δ points due to Balogh, we improve a result of Juhász and Spadaro. A cardinal inequality for linearly Lindelöf Tychonoff spaces proved by Arhangel'skiĭ and Buzyakova should be actually true for Hausdorff spaces. We observe this happens under some restrictions on cardinal arithmetics, including a consequence of Martin's axiom. Finally, we address the question to estimate the cardinality of a first countable linearly H-closed space.

1. Looking for a common extension

For notation we refer to [15]. $[A]^{\leq\kappa} = \{S : S \subseteq A, |S| \leq \kappa\}$. $\mathfrak{c} = 2^\omega$ is the continuum.

$\psi(X)$, $d(X)$, $t(X)$ and $L(X)$ denote pseudocharacter, density, tightness and Lindelöf degree of the topological space X.

A set D in a space X is a free sequence if there is a bijection $f : \kappa \to D$ for some cardinal κ such that $\overline{f([0,\alpha[)} \cap \overline{f([\alpha,\kappa[)} = \emptyset$ for every $\alpha < \kappa$. Given a set $Y \subseteq X$ we denote by $\mathcal{F}(Y)$ the collection of all free sequences in X which are contained in Y. Moreover, $F(X) = \sup\{|D| : D \in \mathcal{F}(X)\}$.

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We always have $F(X) \leq L(X)t(X)$.

A few years ago, Spadaro [20] (and independently Juhász) found a generalization of the well-known Arhangel'skiĭ–Šapirovskiĭ's inequality $|X| \leq 2^{L(X)t(X)\psi(X)}$ (see also [8,9] and [12]).

Proposition 1.1. *[20] If X is a T_2 space, then $|X| \leq 2^{L(X)F(X)\psi(X)}$.*

Much earlier, Arhangel'skiĭ [2] proved that if X is a Lindelöf T_2 sequential space satisfying $\psi(X) \leq \mathfrak{c}$, then $|X| \leq \mathfrak{c}$. A quick look at the proof suffices to realize that this result can actually have the following more general form:

Proposition 1.2. *If X is a T_2 sequential space, then*

$$|X| \leq \psi(X)^{L(X)} = \psi(X)^{L(X)F(X)}.$$

One could be tempted to conjecture that $\psi(X)^{L(X)F(X)}$ is an upper bound for the cardinality of a T_2 space X. Fedorchuk's compact hereditarily separable space of size $2^{\mathfrak{c}}$ immediately shows that this is not the case. The obstacle in Fedorchuk's space is that the cardinality of the closure of a countable set can jump up by more than one exponent. Taking care of this point, we will exhibit in Theorem 1.4 below the right common extension of Propositions 1.1 and 1.2.

Juhász and Nyikos in [17] called a space X *tame* if the inequality $|\overline{A}| \leq 2^{|A|}$ holds for every $A \subseteq X$. We will say that X is κ-tame if $|\overline{A}| \leq 2^\kappa$ whenever $|A| \leq \kappa$. A T_2 sequential space is obviously tame.

The proof of the next theorem mimics an argument used in [12].

If X is a space, κ a cardinal and $Y \subseteq X$, then the κ-closure of Y is the set $[Y]_\kappa = \bigcup\{\overline{A} : A \in [Y]^{\leq\kappa}\}$. Y is κ-closed if $Y = [Y]_\kappa$. Any set of the form $[Y]_\kappa$ is κ-closed.

Lemma 1.3. *[12] Let X be a space and κ an infinite cardinal. If $L(X)F(X) \leq \kappa$ and Y is a κ-closed subspace of X, then $L(Y) \leq \kappa$.*

Proof. Assume by contradiction that $L(Y) > \kappa$ and fix a collection $\mathcal{U}$ of open sets of X such that $Y \subseteq \bigcup\mathcal{U}$ but no $\mathcal{V} \in [\mathcal{U}]^{\leq\kappa}$ satisfies $Y \subseteq \bigcup\mathcal{V}$. Since $L(X) \leq \kappa$ and Y is κ-closed, for each $\alpha \in \kappa^+$ we may select points $x_\alpha \in Y$ and families $\mathcal{V}_\alpha \in [\mathcal{U}]^{\leq\kappa}$ in such a way that:

1) if $\beta < \alpha$ then $\mathcal{V}_\beta \subseteq \mathcal{V}_\alpha$;
2) $\overline{\{x_\beta : \beta < \alpha\}} \subseteq \bigcup\mathcal{V}_\alpha$;
3) $x_\alpha \notin \bigcup\mathcal{V}_\alpha$.

Assume to have already defined x_β and $\mathcal{V}_\beta$ for any $\beta < \alpha$. Being a subset of Y, the set $\overline{\{x_\beta : \beta < \alpha\}}$ is covered by $\mathcal{U}$. Since $L(X) \leq \kappa$ there is some $\mathcal{V}' \in [\mathcal{U}]^{\leq\kappa}$ such that $\overline{\{x_\beta : \beta < \alpha\}} \subseteq \bigcup\mathcal{V}'$. Then, put $\mathcal{V}_\alpha = \mathcal{V}' \cup \bigcup\{\mathcal{V}_\beta : \beta < \alpha\}$. Finally, as $|\mathcal{V}_\alpha| \leq \kappa$, we may pick a point $x_\alpha \in Y \setminus \bigcup\mathcal{V}_\alpha$.

At the end of the induction, we get a free sequence $\{x_\alpha : \alpha < \kappa^+\}$, in contrast with our hypothesis. □

Theorem 1.4. *Let X be a T_1 space and κ an infinite cardinal. If $L(X)F(X) = \kappa$, $\psi(X) \leq 2^\kappa$ and X is κ-tame, then $|X| \leq 2^\kappa$.*

Proof. For any $p \in X$ fix a family $\mathcal{U}(p)$ of open sets satisfying $|\mathcal{U}(p)| \leq 2^\kappa$ and $\{p\} = \bigcap\mathcal{U}(p)$. We plan to define by transfinite induction for each $\alpha < \kappa^+$ a κ-closed set F_α such that:

1_α) if $\beta < \alpha$ then $F_\beta \subseteq F_\alpha$;
2_α) $|F_\alpha| \leq 2^\kappa$;
3_α) if $X \setminus \bigcup\mathcal{V} \neq \emptyset$ for some $\mathcal{V} \in [\bigcup\{\mathcal{U}(p) : p \in F_\alpha\}]^{\leq\kappa}$, then $F_{\alpha+1} \setminus \bigcup\mathcal{V} \neq \emptyset$.