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Meridional rank of knots whose exterior is a graph manifold

Michel Boileau [a,1], Ederson Dutra [b,2], Yeonhee Jang [c], Richard Weidmann [b,*]

[a] *Aix-Marseille Université, CNRS, Centrale Marseille, I2M, UMR 7373, 13453 Marseille, France*
[b] *Mathematisches Seminar, Christian-Albrechts-Universität zu Kiel, Ludewig-Meyn Str. 4, 24098 Kiel, Germany*
[c] *Department of Mathematics, Nara Women's University, Nara, 630-8506, Japan*

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We prove for a large class of knots that the meridional rank coincides with the bridge number. This class contains all knots whose exterior is a graph manifold. This gives a partial answer to a question of S. Cappell and J. Shaneson [10, pb 1.11].

1. Introduction

Let $\mathfrak{k}$ be a knot in S^3. It is well-known that the knot group of $\mathfrak{k}$ can be generated by $b(\mathfrak{k})$ conjugates of the meridian where $b(\mathfrak{k})$ is the bridge number of $\mathfrak{k}$. The meridional rank $w(\mathfrak{k})$ of $\mathfrak{k}$ is the smallest number of conjugates of the meridian that generate its group. Thus we always have $w(\mathfrak{k}) \leq b(\mathfrak{k})$. It was asked by S. Cappell and J. Shaneson [10, pb 1.11], as well as by K. Murasugi, whether the opposite inequality always holds, i.e. whether $b(\mathfrak{k}) = w(\mathfrak{k})$ for any knot $\mathfrak{k}$. To this day no counterexamples are known but the equality has been verified in a number of cases:

(1) For generalized Montesinos knots this is due to Boileau and Zieschang [5].
(2) For torus knots this is a result of Rost and Zieschang [14].
(3) The case of knots of meridional rank 2 (and therefore also knots with bridge number 2) is due to Boileau and Zimmermann [6].
(4) For a class of knots also referred to as generalized Montesinos knots, the equality is due to Lustig and Moriah [11].

* Corresponding author.
E-mail addresses: michel.boileau@univ-amu.fr (M. Boileau), dutra@math.uni-kiel.de (E. Dutra), yeonheejang@cc.nara-wu.ac.jp (Y. Jang), weidmann@math.uni-kiel.de (R. Weidmann).

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(5) For some iterated cable knots this is due to Cornwell and Hemminger [8].
(6) For knots of meridional rank 3 whose double branched cover is a graph manifold the equality can be found in [4].

The *knot space* of $\mathfrak{k}$ is defined as $X(\mathfrak{k}) := \overline{S^3 - V(\mathfrak{k})}$ where $V(\mathfrak{k})$ is a regular neighborhood of $\mathfrak{k}$ in S^3. The *knot group* $G(\mathfrak{k})$ of $\mathfrak{k}$ is the fundamental group of $X(\mathfrak{k})$. We further denote by $P(\mathfrak{k}) \leq G(\mathfrak{k})$ the peripheral subgroup of $\mathfrak{k}$, that is, $P(\mathfrak{k}) = \pi_1 \partial X(\mathfrak{k})$.

Let $m \in P(\mathfrak{k})$ be the meridian of $\mathfrak{k}$, i.e. an element of $G(\mathfrak{k})$ which can be represented by a simple closed curve on $\partial X(\mathfrak{k})$ that bounds a disk in S^3 which intersects $\mathfrak{k}$ in exactly one point, see [7] for more details. In the sequel we refer to any conjugate of m as a meridian of $\mathfrak{k}$.

We call a subgroup $U \leq G(\mathfrak{k})$ *meridional* if U is generated by finitely many meridians of $\mathfrak{k}$. The minimal number of meridians needed to generated U, denoted by $w(U)$, is called the *meridional rank* of U. Observe that the knot group $G(\mathfrak{k})$ is meridional and its meridional rank is equal to $w(\mathfrak{k})$.

A meridional subgroup U of meridional rank $w(U) = l$ is called *tame* if for any $g \in G(\mathfrak{k})$ one of the following holds:

(1) $gP(\mathfrak{k})g^{-1} \cap U = 1$.
(2) $gP(\mathfrak{k})g^{-1} \cap U = g\langle m \rangle g^{-1}$ and there exist meridians $m'_2, \ldots, m'_l$ such that U is generated by $\{gmg^{-1}, m'_2, \ldots, m'_l\}$.

Definition 1.1. A non-trivial knot $\mathfrak{k}$ in S^3 is called *meridionally tame* if any meridional subgroup $U \leq G(\mathfrak{k})$ generated by less than $b(\mathfrak{k})$ meridians is tame.

Remark 1.2. If $\mathfrak{k}$ is meridionally tame, then its group cannot be generated by less than $b(\mathfrak{k})$ meridians. Hence the bridge number equals the meridional rank. Thus the question of Cappell and Shaneson has a positive answer for the class of meridionally tame knots by definition of meridional tameness.

The class of meridionally tame knots trivially contains the class of 2-bridge knots as any cyclic meridional subgroup is obviously tame. In Lemma 6.1 below we show that the meridional tameness of torus knots is implicit in [14] and in Proposition 7.1 we show that prime 3-bridge knots are meridionally tame. However it follows from [1] and the discussion at the end of Section 7 that satellite knots are in general not meridionally tame. For examples Whitehead doubles of non-trivial knots are never meridionally tame. These are prime satellite knots, whose satellite patterns have winding number zero, which is opposite to the braid patterns considered in this article. Moreover Whitehead doubles of 2-bridge knots are prime 4-bridge knots for which the question of Cappell and Shaneson has a positive answer by [4, Corollary 1.6]. In contrast, it should be noted that we do not know any hyperbolic knots that are not meridionally tame, but it is likely that such knots exist.

In this article we consider the class of knots $\mathcal{K}$ that is the smallest class of knots that contains all meridionally tame knots and is closed under connected sums and satellites with braid pattern, see Section 2 for details. The following result is our main theorem:

Theorem 1.3. *Let $\mathfrak{k}$ be a knot from $\mathcal{K}$. Then $w(\mathfrak{k}) = b(\mathfrak{k})$.*

As the only Seifert-fibered manifolds that can be embedded into a knot manifold with incompressible boundary are torus knot complements, composing spaces and cable spaces (see Lemma VI.3.4 of [12]) and as cable spaces are special instances of braid patterns we immediately obtain the following consequence of Theorem 1.3.