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Chromatic homology, Khovanov homology, and torsion

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ABSTRACT

In the first few homological gradings, there is an isomorphism between the Khovanov homology of a link and the categorification of the chromatic polynomial of a graph related to the link. In this article, we show that all torsion in the categorification of the chromatic polynomial is of order two, and hence all torsion in Khovanov homology in the gradings where the isomorphism is defined is of order two. We also prove that odd Khovanov homology is torsion-free in its first few homological gradings.

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1. Introduction

Khovanov homology is a categorification of the Jones polynomial constructed by Khovanov in [11]. The Khovanov homology $Kh(L)$ of a link L is a finitely generated, bigraded abelian group. Experimental computations show that Khovanov homology frequently has nontrivial torsion. Torsion of order two has been studied extensively by Shumakovitch [20], Asaeda and Przytycki [1], Pabinak, Przytycki, and Sazdanović [17], and Przytycki and Sazdanović [18]. Shumakovitch conjectures that the Khovanov homology of every link except disjoint unions and connected sums of unknots and Hopf links contains torsion of order two. This conjecture has been confirmed for many special cases, including alternating links and many semi-adequate links.

Much less is known about torsion of order not equal to two. Computations by Bar-Natan [3] show that the Khovanov homology of the $(4, q)$ torus knot for $q = 5, 7, 9$, or 11 contains torsion of order four. Further computer computations by Bar-Natan and Green [4] show that torus knots of higher braid index can have odd torsion. In a recent paper [15], Mukherjee, Przytycki, Silvero, Wang, and Yang give many more examples of links whose Khovanov homology contains torsion of odd order. Experimental computations show that

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Fig. 1. Positive and negative crossings in a link diagram.

among knots with few crossings, knots with torsion of orders other than two in their Khovanov homology are less common than knots whose Khovanov homology contains only torsion of order two.

Theorem 1.1 gives a partial explanation to this observation, at least in the first few and last few homological gradings of Khovanov homology. The *all-A state graph* $G_A(D)$ of D is a graph obtained from the all-A Kauffman state of D. See Section 2 for more on Kauffman states and Section 5 for the precise construction of state graphs. The *girth* of $G_A(D)$ is the length of the shortest cycle in $G_A(D)$. See Fig. 1 for our conventions on positive and negative crossings.

Theorem 1.1. *Let D be a link diagram with c_- negative crossings such that its all-A state graph $G_A(D)$ has girth g of at least two. If $-c_- \leq i \leq -c_- + g - 1$, then all torsion in $Kh^{i,j}(D)$ is of order two.*

Since $Kh^{i,j}(D) = 0$ when $i < -c_-$, the gradings where Theorem 1.1 apply are the first few homological gradings where the Khovanov homology is nonzero. An analogous statement holds for the last few homological gradings of $Kh(D)$. Let g' be the girth of the all-A state graph of the mirror of D (or equivalently of the all-B state graph of D). Theorem 1.1 and the relationship between the Khovanov homology of a link and its mirror then implies that all torsion in $Kh^{i,j}(D)$ is of order two when $c_+ - g' + 1 \leq i \leq c_+$, where c_+ is the number of positive crossings in D. Moreover $Kh^{i,j}(D) = 0$ if $i > c_+$.

Ozsváth, Rasmussen, and Szabó [16] define the odd Khovanov homology $Kh_{\text{odd}}(L)$ of a link L, a categorification of the Jones polynomial that agrees with Khovanov homology with $\mathbb{Z}_2$ coefficients, but differs with $\mathbb{Z}$ coefficients. In Section 6, we prove a version of Theorem 1.1 for odd Khovanov homology.

Theorem 1.2. *Let D be a link diagram with c_- negative crossings such that its all-A state graph $G_A(D)$ is planar and has girth g of at least two. If $-c_- \leq i \leq -c_- + g - 1$, then $Kh^{i,j}_{\text{odd}}(D)$ has no torsion.*

Helme-Guizon and Rong [8] define a categorification of the chromatic polynomial of a graph G, which we call the *chromatic homology* of G. One can view this theory as a comultiplication-free version of Khovanov homology. The Khovanov homology of a link and the chromatic homology of the all-A state graph of a diagram of the link are isomorphic in certain bigradings. Theorem 1.1 is a consequence of this relationship together with the following result on chromatic homology.

Theorem 1.3. *All torsion in the chromatic homology of a graph is of order two.*

Theorem 1.3 is proved in two major steps. First, we show that chromatic homology cannot have torsion of odd order by examining a version of Lee's spectral sequence [13] for chromatic homology with $\mathbb{Z}_p$ coefficients where p is an odd prime. Next, we prove that the only torsion of order 2^k in chromatic homology is in fact of order two. In order to achieve this second step, we define new maps $\nu_\downarrow$, $\nu_\uparrow$, and d_T on the chromatic complex with $\mathbb{Z}_2$ coefficients, each of which induces a map on chromatic homology with $\mathbb{Z}_2$ coefficients. These maps are inspired by similar maps on the Khovanov complex with $\mathbb{Z}_2$ coefficients defined by Shumakovitch [20] and Turner [23]. We relate the induced maps to the differential in the $\mathbb{Z}_2$-Bockstein spectral sequence for chromatic homology to prove the desired result.

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