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On extension functions for image space with different separation axioms ☆

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ABSTRACT

In this paper we study a sufficient conditions for continuous and θ_α-continuous extensions of f to space X for an image space Y with different separation axioms.

1. Introduction

This paper is devoted to a systematic study of the general problem, which is as follows. Let f be a continuous mapping of a dense set S of the topological space X into the topological space Y. Required to find the necessary and sufficient conditions for continuous extension of f to the space X (i.e. existence a continuous mapping $F : X \mapsto Y$ such that $F \upharpoonright S = f$). This problem can be considered more widely, if the continuous mapping is replaced be "almost" continuous. For example, we will consider the θ_α-continuous mapping.

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First sufficient condition for continuous extension of f to the space X into the regular space Y was obtained by N. Bourbaki. In [5] was proved that this condition is not sufficient condition for no regular space Y.

The necessary and sufficient conditions for continuous extension of f on the space X were obtained:

in [6] for metrizable compact spaces Y;

in [4] for compact spaces Y;

in [3] for Lindelëof spaces Y;

in [8] for realcompact spaces Y;

in [3] for regular spaces Y.

So for a compact spaces Y we have the next result (see [4]).

Theorem 1.1 *(Taimanov). Let f be a continuous mapping of a dense set S of a topological space X into a compact space Y, then the following are equivalent:*

1. *f to have a continuous extension to X;*
2. *if A and B are disjoint closed subsets of Y then $\overline{f^{-1}(A)} \bigcap \overline{f^{-1}(B)} = \emptyset$.*

Consider the following

- Condition $(*)$: if a family $\{A_\beta\}$ of closed subsets of Y such that $\bigcap_\beta A_\beta = \emptyset$ implies $\bigcap_\beta \overline{f^{-1}(A_\beta)} = \emptyset$.

So general result for a regular space Y is the following theorem [3].

Theorem 1.2 *(Velichko). Let f be a continuous mapping of a dense set S of a topological space X into a regular space Y, then the following are equivalent:*

1. *f to have a continuous extension to X;*
2. *condition $(*)$ holds.*

Note that if Y is a Tychonoff space, then we have in condition $(*)$ a closed subsets be replaced by zero-sets of Y.

Theorem 1.3 *(Velichko). Let f be a continuous mapping of a dense set S of the topological space X into a Lindelëof space Y, then the following are equivalent:*

1. *f to have a continuous extension to X;*
2. *for any sequence $\{A_i\}$ of zero-sets of Y such that $\bigcap_i A_i = \emptyset$ implies $\bigcap_i \overline{f^{-1}(A_i)} = \emptyset$.*

Note that the condition $(*)$ is a necessary condition for continuous extension of f to X for any space Y.

Proposition 1.4. *Let f have a continuous extension to X for a space Y. Then condition $(*)$ holds.*

Proof. Let F be a continuous extension to X for a space Y and $\{A_\beta\}$ be a family of closed subsets of Y such that $\bigcap_\beta A_\beta = \emptyset$. Fix $x \in X$. There is β such that $F(x) \notin A_\beta$, hence there exists a neighborhood V of $F(x)$ such that $V \bigcap A_\beta = \emptyset$. Since F is continuous map, $F^{-1}(V)$ is a neighborhood of x. It follows that $F^{-1}(V) \bigcap F^{-1}(A_\beta) = \emptyset$ and $x \notin \overline{f^{-1}(A_\beta)}$. □