Topology and its Applications 225 (2017) 46–52

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Virtual Special Issue – Proceedings on the International Conference on Set-Theoretic Topology and its Applications, Yokohama 2015

Finite-valued multiselections

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ARTICLE INFO

Article history:
Received 22 December 2015
Received in revised form 24 May 2016
Accepted 27 May 2016
Available online 21 April 2017

MSC:
54B20
54C60
54C65
54D05
54F05

Keywords:
Vietoris topology
Multiselection
Weak selection
Weak orderability

ABSTRACT

It is shown that a connected space X is weakly orderable provided it has a finite-valued Vietoris continuous multiselection for its hyperspace $\mathscr{F}(X)$ of nonempty closed subsets. In fact, for connected spaces, every such multiselection is at most two-point valued, and X is compact whenever the multiselection is not singleton-valued at some element of $\mathscr{F}(X)$. Complementary to this result is a characterisation of weak orderability of connected spaces in terms of "proper" Vietoris continuous multiselections for hyperspaces of finite sets.

1. Introduction

All spaces are assumed to be infinite Hausdorff topological spaces. Let $\mathscr{F}(X)$ be the collection of all nonempty closed subsets of a space X. Each subcollection $\mathscr{D} \subset \mathscr{F}(X)$ will carry the (relative) *Vietoris topology* τ_V, and will be simply called a *hyperspace*. The basic τ_V-neighbourhoods for this topology on $\mathscr{D}$ are the sets

$$\langle \mathscr{V} \rangle = \left\{ S \in \mathscr{D} : S \subset \bigcup \mathscr{V} \text{ and } S \cap V \neq \varnothing, \text{ whenever } V \in \mathscr{V} \right\},$$

where $\mathscr{V}$ runs over the finite families of open subsets of X.

For spaces Z and X, a set-valued mapping $\varphi : Z \to \mathscr{F}(X)$ is *lower semi-continuous*, or *l.s.c.*, if the set

$$\varphi^{-1}[U] = \{z \in Z : \varphi(z) \cap U \neq \varnothing\}$$

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http://dx.doi.org/10.1016/j.topol.2017.04.017

is open in Z for every open $U \subset X$; and φ is *upper semi-continuous*, or *u.s.c.*, if $\varphi^{-1}[F]$ is closed in Z for every closed $F \subset X$. A mapping $\varphi : Z \to \mathscr{F}(X)$ is called *continuous* (sometimes also *Vietoris continuous*, or τ_V*-continuous*) if it is both l.s.c. and u.s.c. Accordingly, $\varphi : Z \to \mathscr{F}(X)$ is continuous if and only if it is continuous as a usual map from Z to the hyperspace $(\mathscr{F}(X), \tau_V)$.

A map $f : \mathscr{D} \to X$ is a *selection* for $\mathscr{D} \subset \mathscr{F}(X)$ if $f(S) \in S$ for every $S \in \mathscr{D}$, and f is *continuous* if it is continuous with respect to the relative Vietoris topology τ_V on $\mathscr{D}$. In the present paper, we are interested in set-valued selections for hyperspaces. We shall say that a mapping $\varphi : \mathscr{D} \to \mathscr{F}(X)$ is a *multiselection* (or a *set-valued selection*) for $\mathscr{D} \subset \mathscr{F}(X)$ if $\varphi(S) \subset S$ for every $S \in \mathscr{D}$. In these terms, we say that φ is τ_V*-l.s.c.* (respectively, τ_V*-u.s.c.*) if it is l.s.c. (respectively, u.s.c.) as a set-valued mapping from the hyperspace $(\mathscr{D}, \tau_V)$ to $\mathscr{F}(X)$. A multiselection $\varphi : \mathscr{D} \to \mathscr{F}(X)$ which is both τ_V-l.s.c. and τ_V-u.s.c. will be called τ_V*-continuous* being a continuous map from $(\mathscr{D}, \tau_V)$ to $(\mathscr{F}(X), \tau_V)$. It should be remarked that $\mathscr{F}(X)$ always has a τ_V-continuous multiselection, one can merely take the identity of $\mathscr{F}(X)$. However, if extra properties on the values of the multiselection are required, then the corresponding selection problem may characterise certain spaces. For instance, each continuous selection $f : \mathscr{F}(X) \to X$ can be considered as a τ_V-continuous multiselection by letting $\varphi(S) = \{f(S)\}$, $S \in \mathscr{F}(X)$. Let

$$\mathscr{C}(X) = \{S \in \mathscr{F}(X) : S \text{ is compact}\}.$$

Compact-valued multiselections have been useful in showing that a metrizable space X is completely metrizable if and only if it has a τ_V-u.s.c. multiselection $\theta : \mathscr{F}(X) \to \mathscr{C}(X)$, see [10, Theorem 7.1] and [3, Corollary 7.5]. For some open questions on τ_V-continuous multiselections for hyperspaces on compact scattered spaces, the interested reader is referred to [4], see also [5].

A space X is *orderable* (or *linearly ordered*) if it has the open interval topology generated by some linear order on X. A space X is *weakly orderable* (*KOTS* in the terminology of [11]; and sometimes called also “Eilenberg orderable”) if it has a coarser open interval topology. Whenever $n \geq 1$, let

$$\mathscr{F}_n(X) = \{S \in \mathscr{F}(X) : |S| \leq n\}.$$

A selection $\sigma : \mathscr{F}_2(X) \to X$ is often called a *weak selection* for X. If X is connected and has a continuous weak selection σ, then it is weakly orderable. In this case, the relation $x \preceq_\sigma y$ defined by $\sigma(\{x, y\}) = x$, is a *compatible* linear order on X, i.e. X is weakly orderable with respect to $\preceq_\sigma$ [8, Lemma 7.2]. However, in general, $\preceq_\sigma$ is not necessarily transitive. Treating selections as singleton-valued multiselections, (continuous) weak selections for X are identical to (τ_V-continuous) multiselections $\varphi : \mathscr{F}_2(X) \to \mathscr{F}_1(X)$. In the presence of connectedness, the role of such multiselections is preserved for arbitrary $n \geq 1$ as our first result asserts.

Theorem 1.1. *Let X be a connected space which has a τ_V-continuous multiselection $\varphi : \mathscr{F}_{n+1}(X) \to \mathscr{F}_n(X)$ for some $n \geq 1$. Then X is weakly orderable.*

The converse of Theorem 1.1 is also true. Namely, if X is weakly orderable, then $\mathscr{F}_{n+1}(X)$ has a continuous selection for every $n \geq 1$ [8, Lemma 7.5.1]; the same is also valid for $\mathscr{C}(X)$. Complementary to Theorem 1.1 is now the following result for the nonempty finite subsets $\Sigma(X) = \bigcup_{n \geq 1} \mathscr{F}_n(X)$.

Theorem 1.2. *Let X be a connected space which has a τ_V-continuous multiselection $\varphi : \mathscr{F}(X) \to \Sigma(X)$. If φ is not singleton-valued at some element of $\mathscr{F}(X)$, then X is compact and orderable.*

The statements of Theorems 1.1 and 1.2 appear to be the best possible in several respects. Here is an example showing that both theorems fail if “τ_V-continuity” of the multiselection is relaxed to “τ_V-semi-continuity”.