Topology and its Applications 225 (2017) 67–74

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Virtual Special Issue – Proceedings on the International Conference on Set-Theoretic Topology and its Applications, Yokohama 2015

Rigidity of embeddings of finite products of certain continua

Stanisław Spież

Institute of Mathematics, Polish Academy of Sciences, ul. Śniadeckich 8, 00-656 Warsaw, Poland

ARTICLE INFO

Article history:
Received 1 February 2016
Received in revised form 30 November 2016
Accepted 30 November 2016
Available online 22 April 2017

MSC:
primary 54C25, 54B10
secondary 54F15, 54F35, 55N05, 55N45

Keywords:
Embeddings
Factorwise rigidity
Čech cohomology
Cup product

ABSTRACT

We show that an embedding $X_1 \times \ldots \times X_n \to Y_1 \times \ldots \times Y_n$ of products k-dimensional continua is factorwise rigid provided each X_i contains a dense family of "cohomological holes" and each Y_j is cohomologically $(k-1)$-connected. As a corollary we obtain that any embedding of $M_k^m \times \ldots \times M_k^m$ into itself, where M_k^m are the Menger spaces with $1 \leq k < m$, is factorwise rigid.

1. Introduction

We say that a function $f : \prod_{a\in A} X_a \to \prod_{b\in B} Y_b$ is *factorwise rigid* (cf., [2] and [6]) if there is a bijection $\varepsilon : A \to B$ and functions $f_a : X_a \to Y_{\varepsilon(a)}$ such that $(f(x))_{\varepsilon(a)} = f_a(x_a)$ for each $x = (x_a)_{a\in A}$ and each $a \in A$.

By a *continuum* we mean a connected compact metric space, and by a *curve* a 1-dimensional continuum.

We say that a curve X contains a *dense family of circles* if each open subset of X contains a simple closed curve. The following result was proved by R. Cauty [1] (see also [8]).

Theorem 1.1. *Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ be locally connected curves with dense families of circles. Then every homeomorphism $X_1 \times \ldots \times X_n \to Y_1 \times \ldots \times Y_n$ is factorwise rigid.*

For embeddings, a result of this type was proved in [2] (Theorem 2.3).

Theorem 1.2. *Every embedding of the product of two pseudo-arcs is factorwise rigid.*

E-mail address: spiez@impan.pl.

http://dx.doi.org/10.1016/j.topol.2017.04.020
0166-8641/

In this note we extend the Cauty's theorem for embeddings between products of certain higher dimensional continua which are not necessarily locally connected, or even arcwise connected.

Below, H denotes the Čech cohomology functor with the integer coefficients.

A continuum X is *cohomologically k-connected* if $H^i(X) = 0$ for all $i \in \{1, \ldots, k\}$. We say that a k-dimensional continuum X contains *a dense family of cohomological holes* if each open subset of X contains a cohomologically $(k-1)$-connected continuum C with $H^k(C) \in \mathcal{G}$, where $\mathcal{G}$ denotes the class of all abelian groups G such that $G \otimes H \neq 0$ for each non-trivial abelian group H. (The class $\mathcal{G}$ contains all finitely generated abelian groups which contain non-torsion elements.)

The main result of this note is the following theorem.

Theorem 1.3. *Let $X_1, \ldots, X_n$ be k-dimensional continua with dense families of cohomological holes and let $Y_1, \ldots, Y_n$ be cohomologically $(k-1)$-connected k-dimensional continua, $k \geq 1$. Then any embedding $X_1 \times \ldots \times X_n \to Y_1 \times \ldots \times Y_n$ is factorwise rigid.*

The proof of this theorem, in Section 2, follows some general ideas of the proof in [1] (also cf., [8]). It involves the cohomology ring of spaces.

In the case of curves we obtain the following corollary which generalizes Theorem 1.1.

Corollary 1.4. *Let $X_1, \ldots, X_n$ be curves with dense families of cohomological holes (e.g., with dense families of circles) and $Y_1, \ldots, Y_n$ arbitrary curves. Then any embedding $X_1 \times \ldots \times X_n \to Y_1 \times \ldots \times Y_n$ is factorwise rigid.*

It is known that any Menger space M_k^m (for the notation see [4]), with $1 \leq k \leq m$, is a cohomologically $(k-1)$-connected[1] k-dimensional continuum and if it is not a k-cube (i.e., if $k < m$) then any open subset of M_k^m contains a continuum homeomorphic to the sphere S^k. Thus we obtain the following corollary.

Corollary 1.5. *Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ be k-dimensional Menger spaces, where each X_i is not a k-cube. Then any embedding $X_1 \times \ldots \times X_n \to Y_1 \times \ldots \times Y_n$ is factorwise rigid.*

2. Proof of the main result – Theorem 1.3

Lemma 2.1. *Let C_1 and C_2 be continua, where C_1 is cohomologically $(k-1)$-connected, and let $\iota_1 : C_1 \times \{c_2\} \to C_1 \times C_2$ and $\iota_2 : \{c_1\} \times C_2 \to C_1 \times C_2$ denote the inclusions, where $c_1 \in C_1$ and $c_2 \in C_2$, and $p_2 : C_1 \times C_2 \to \{c_1\} \times C_2$ the projection. Then $H^k(\iota_2 \circ p_2)(g) = g$ for all $g \in \operatorname{Ker} H^k(\iota_1)$.*

Proof. By the Künneth formula, [11], p. 249, we have the following commutative diagram

$$\begin{array}{ccc} [H^*(C_1) \otimes H^*(C_2)]^k & \longrightarrow & H^k(C_1) \otimes H^0(\{c_2\}) \\ \Big\downarrow \mu & & \Big\downarrow \mu' \\ H^k(C_1 \times C_2) & \xrightarrow[H^k(\iota_1)]{} & H^k(C_1 \times \{c_2\}). \end{array}$$

Since C_1 is cohomologically $(k-1)$-connected, and since $H^0(C)$ and $H^1(C)$ are torsion free for any continuum C, it follows that $[H^*(C_1) * H^*(C_2)]^{k+1} = 0$, so by the Künneth formula μ is an isomorphism. Note that also μ' is an isomorphism.

[1] Since it is homotopically $(k-1)$-connected.