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A note on the existence of essential tribranched surfaces

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A R T I C L E   I N F O

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A B S T R A C T

The second author and Hara introduced the notion of an essential tribranched surface that is a generalisation of the notion of an essential embedded surface in a 3-manifold. We show that any 3-manifold for which the fundamental group has at least rank four admits an essential tribranched surface.

1. Introduction

Let M be a 3-manifold. Here and throughout the paper all 3-manifolds are assumed to be compact, connected and orientable. A properly embedded, connected, orientable surface $\Sigma \neq S^2$ in M is called *essential* if the inclusion induced map $\pi_1(\Sigma) \to \pi_1(N)$ is a monomorphism and if Σ is not boundary-parallel. A 3-manifold is called *Haken* if it is irreducible and if it admits an essential surface.

Haken manifolds play a major role in 3-manifold topology. For example, Waldhausen [12] gave a solution to the homeomorphism problem for Haken 3-manifolds, Thurston [11] proved the Geometrisation Theorem for Haken 3-manifolds and recently Wise [13] showed that Haken hyperbolic 3-manifolds are virtually fibred. The principle in each case is that the existence of an essential (in [13] geometrically finite) surface leads to a hierarchy, which allows a proof by induction.

It is well-known and straightforward to show that any irreducible 3-manifold M with $b_1(M) \geq 1$ is Haken. This implies in particular that any irreducible 3-manifold $M \neq D^3$ with non-empty boundary is Haken. A particularly interesting source of essential surfaces in a 3-manifold M is provided by the work Culler and Shalen [4]. They showed how ideal points of the character variety $\mathrm{Hom}(\pi_1(M), \mathrm{SL}(2,\mathbb{C}))//\,\mathrm{SL}(2,\mathbb{C})$ give rise to essential surfaces in M.

Despite the abundance of Haken manifolds, there are many examples of closed, irreducible 3-manifolds that are not Haken. For example most Dehn surgeries along the Figure-8 knot are hyperbolic but not Haken.

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We refer to [6,10,2] and also the discussion in [1, (C.17)] for details and many more examples. It is still an open, and very interesting question, whether or not a 'generic' 3-manifold is Haken.

Recently Hara and the second author [7] generalised the work of Culler and Shalen [4] to higher dimensional character varieties. More precisely, let M be an irreducible 3-manifold with non-empty boundary and let $n \geq 3$ be a natural number. It is shown in [7] that any ideal point of the character variety $\mathrm{Hom}(\pi_1(M), \mathrm{SL}(n,\mathbb{C}))//\mathrm{SL}(n,\mathbb{C})$ gives rise to an essential *tribranched* surface. For $n = 3$ this even holds if M has no boundary. We will recall the definition of an essential tribranched surface in Section 2. For the moment it suffices to know that an essential surface is also an essential tribranched surface, but as the name already suggests, tribranched surfaces are allowed to have a certain controlled type of branching.

The following question was raised in [7, Question 6.5].

Question 1.1. Does every aspherical 3-manifold contain an essential tribranched surface?

The goal of this paper is to give an affirmative answer for 'most' 3-manifolds. More precisely, given a finitely generated group π we denote by $\operatorname{rank} \pi$ the minimal number of generators of π. The following is our main theorem.

Theorem 1.2. *Let M be a closed 3-manifold with $\operatorname{rank} \pi_1(M) \geq 4$. Then M admits an essential tribranched surface.*

The importance to find an essential tribranched surface in M is that it provides a nontrivial presentation of $\pi_1(M)$ in terms of a certain 2-dimensional complex of groups as in [7, Proposition 2.5]. It can been seen as an analogous extension of the fact that an essential surface in M induces a nontrivial splitting of $\pi_1(M)$ as the fundamental group of a graph of groups. Thus Theorem 1.2 implies that if $\operatorname{rank} \pi_1(M) \geq 4$, then $\pi_1(M)$ admits such a weaker kind of splitting even for a non-Haken manifold M.

We will construct these tribranched surfaces using open book decompositions. In particular, our approach is quite different from the approach taken in [7].

Conventions. All 3-manifolds are assumed to be compact, connected and orientable, unless it says explicitly otherwise. Furthermore all self-diffeomorphisms of surfaces are assumed to be orientation preserving. We identify $\mathbb{R}/\mathbb{Z}$ with S^1 via the map $t \mapsto e^{2\pi i t}$.

2. Tribranched surfaces

We recall the definition of a tribranched surface from [7]. In this section M will always denote a closed 3-manifold.

Definition 2.1. Let $\Sigma \subset M$ be a subset and X another 3-manifold.

(1) For a subset Λ of X we say that an open set U of M is *modelled* on $\Lambda \subset X$ if there exists an open set $V \subset X$ and a homeomorphism $\phi\colon U \to V$ with $\phi(\Sigma \cap U) = \Lambda \cap V$.
(2) For a collection $\mathcal{A}$ of subsets of X we say Σ is *locally modelled* on $\mathcal{A}$ if given any point $x \in \Sigma$ there exist an open neighbourhood U_x of x in M and a subset $\Lambda_x \in \mathcal{A}$ such that U_x is modelled on $\Lambda_x \subset X$.

Example 2.2. A 2-dimensional submanifold of a 3-manifold is a subset locally modelled on $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$.

Definition 2.3.

(1) A *pre-tribranched surface* Σ in M is a closed subset of M locally modelled on the collection of the following subsets: