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Thirty-two equivalence relations on knot projections

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We consider 32 homotopy classifications of knot projections (images of generic immersions from a circle into a 2-sphere). These 32 equivalence relations are obtained based on which moves are forbidden among the five types of Reidemeister moves. We show that 32 cases contain 20 non-trivial cases that are mutually different. To complete the proof, we obtain new tools, i.e., new invariants.

1. Introduction

A *knot projection* is the image of a generic immersion from a circle into a 2-sphere. In particular, every self-intersection is a transverse double point, which is simply called a *double point*. Every double point consists of two branches and thus, if the two branches are given over/under information for every double point of a knot projection, we can then obtain a *knot diagram*.

Several interesting homotopy classes have been considered by restricting the *Reidemeister moves* that consist of three types of local replacements of knot projections, as shown in Fig. 1 [1,2,5,7–11] (for other works, see [4]). Because a knot projection is a single component, we consider five types of Reidemeister moves, namely, RI, strong RII, weak RII, strong RIII, and weak RIII, which are the local replacements defined in Fig. 2.

We can consider 32 ($= 2^5$) equivalence classes by restricting the Reidemeister moves. Naturally, we have Problem 1 as below.

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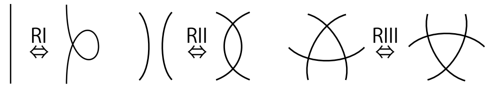

Fig. 1. Reidemeister moves.

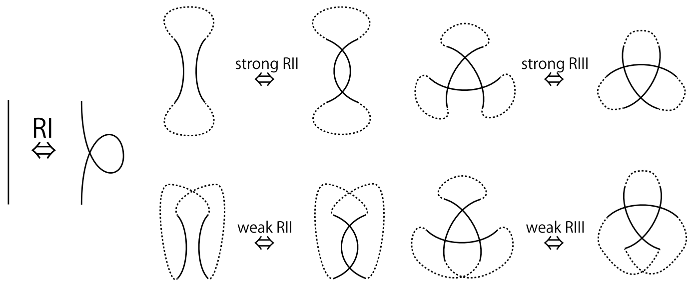

Fig. 2. RI, strong RII, weak RII, strong RIII, weak RIII. Each dotted arc indicates the connection between branches.

Problem 1. (1) Which equivalence classes of knot projections are non-trivial?
(2) Which two equivalence relations on knot projections are independent?

Theorem 1 described in this paper solves Problem 1.

2. Main result

Definition 1. Let $\mathcal{R} = \{\text{RI}, \text{strong RII}, \text{weak RII}, \text{strong RIII}, \text{weak RIII}\}$ and let $\mathcal{S}$ be a subset of $\mathcal{R}$. We say that two knot projections P and P' are $\mathcal{S}$-*equivalent* if they can be related by a finite sequence consisting of the elements of $\mathcal{S}$. We denote this equivalence by $\sim_{\mathcal{S}}$. There are 32 $(= 2^5)$ possibilities of type

$$\{\text{knot projections}\}/\sim_{\mathcal{S}}$$

that is denoted by $\mathcal{C}_{\mathcal{S}}$.

Theorem 1. *Let $\mathcal{C}_{\mathcal{S}}$ be as defined in Definition 1. Among the 32 possible sets, 8 sets are trivial (i.e., all knot projections are equivalent to the trivial knot projection) and the remaining 24 sets are equivalent to the following 20 sets that are non-trivial and mutually different:*
$\mathcal{S}$ = {strong RII, weak RII, strong RIII, weak RIII}, {strong RII, strong RIII, weak RIII}, {weak RII, strong RIII}, {weak RII, strong RIII, weak RIII}, {strong RII}, {weak RII}, {strong RII, weak RII}, {weak RII, weak RIII}, {strong RIII, weak RIII}, {strong RIII}, {weak RIII}, {RI}, {RI, strong RII}, {RI, weak RII}, {RI, strong RII, weak RII}, {RI, strong RIII}, {RI, weak RIII}, {RI, strong RIII, weak RIII}, {RI, weak RII, weak RIII}, or $\emptyset$.

3. Invariants of knot projections

3.1. New tools—new invariants

In this section, we introduce a new invariant $\text{Coh}^{\text{odd}}(P)$ for a knot projection P under weak RII and strong RIII to detect one of the two cases: $\mathcal{C}_{\{\text{weak RII, strong RIII}\}}$ and $\mathcal{C}_{\{\text{weak RII, strong RIII, weak RIII}\}}$.

Let P be a knot projection with an arbitrary orientation. If the orientation induces an orientation of an n-gon of P (i.e., the orientations of n edges are coherent), the n-gon is called a *coherent* n-gon. An n-gon that is not coherent is called an *incoherent* n-gon. The sum of the number of coherent $(2m+1)$-gons $(m \in \mathbb{Z}_{\geq 0})$ is called the *odd coherent number*. We set the function Coh^{odd} from the set of knot projections to $\{0, 1\}$ such that