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A Golod complex with non-suspension moment-angle complex

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A B S T R A C T

It could be expected that the moment-angle complex associated with a Golod simplicial complex is homotopy equivalent to a suspension space. In this paper, we provide a counter example to this expectation. We have discovered this complex through the studies of the Golod property of the Alexander dual of a join of simplicial complexes, and that of a union of simplicial complexes.

1. Introduction

The *Stanley–Reisner ring* (or *face ring*) of a simplicial complex K over an index set $[m] = \{1, \cdots, m\}$ is defined as the quotient graded algebra $\mathbf{k}[K] = \mathbf{k}[v_1, \cdots, v_m]/\mathcal{I}_K$, where $\mathbf{k}$ is a commutative ring with unit and $\mathcal{I}_K = (v_{i_1} \cdots v_{i_k} \mid \{i_1, \cdots, i_k\} \notin K)$ is the *Stanley–Reisner ideal* of K. K is called *Golod* over a field $\mathbf{k}$ if its Stanley–Reisner ring $\mathbf{k}[K]$ is Golod over $\mathbf{k}$. That is, the multiplication and all higher Massey products in

$$\mathrm{Tor}^{\mathbf{k}[v_1,\cdots,v_m]}(\mathbf{k}[K], \mathbf{k}) = H(\Lambda[u_1, \cdots, u_m] \otimes \mathbf{k}[K], d)$$

are trivial, where the Koszul differential algebra $(\Lambda[u_1, \cdots, u_m] \otimes \mathbf{k}[K], d)$ is the bigraded differential algebra with $\deg u_i = (1, 2)$, $\deg v_i = (0, 2)$, and $du_i = v_i$ for $i = 1, \cdots, m$. Originally, the algebra $\mathbf{k}[K]$ or the ideal $\mathcal{I}_K$ was defined to be Golod if the following equation holds:

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$$\sum_{i\geq 0;\ j\geq 0} \dim_{\mathbf{k}} \mathrm{Tor}_{j,2i}^{\mathbf{k}[K]}(\mathbf{k},\mathbf{k}) t^j z^i = \frac{(1+tz)^m}{1 - t\sum_{i\geq 0;\ j\geq 1} \dim_{\mathbf{k}} \mathrm{Tor}_{j,2i}^{\mathbf{k}[v_1,\cdots,v_m]}(\mathbf{k}[K],\mathbf{k}) t^j z^i},$$

where $\mathrm{Tor}_{j,2i}^{\mathbf{k}[K]}(\mathbf{k},\mathbf{k})$ and $\mathrm{Tor}_{j,2i}^{\mathbf{k}[v_1,\cdots,v_m]}(\mathbf{k}[K],\mathbf{k})$ denote the homogeneous components of degree $2i$. Golod [8] proved the equivalence of the two conditions, and thereafter his name has been used to refer a ring that satisfies the condition. The reader may also refer to Gulliksen and Levin [11] or Avramov [1].

Baskakov, Buchstaber, and Panov [3] and Franz [7] independently demonstrated that the torsion algebra $\mathrm{Tor}^{\mathbf{k}[v_1,\cdots,v_m]}(\mathbf{k}[K],\mathbf{k})$ is isomorphic to the cohomology ring of the moment-angle complex Z_K associated with K.

Theorem 1.1 *([3,7]). For a commutative ring* $\mathbf{k}$ *with unit, the following isomorphisms of algebras hold:*

$$H^*(Z_K;\mathbf{k}) \cong \mathrm{Tor}_*^{\mathbf{k}[v_1,\cdots,v_m]}(\mathbf{k}[K],\mathbf{k}) \cong \bigoplus_{I\subset[m]} \tilde{H}^*(K_I;\mathbf{k}),$$

where $\tilde{H}^*(K_I;\mathbf{k})$ *denotes the reduced cohomology of the full subcomplex* K_I *of* K *on* I*, and* $\tilde{H}^*(K_\emptyset;\mathbf{k}) = 0$ *for* $* \neq -1$ *and* $= \mathbf{k}$ *for* $* = -1$*. The last isomorphism is the sum of isomorphisms given by*

$$H^p(Z_K;\mathbf{k}) \cong \bigoplus_{I\subset[m]} \tilde{H}^{p-|I|-1}(K_I;\mathbf{k}),$$

and the ring structure is given by the maps

$$\tilde{H}^{p-|I|-1}(K_I;\mathbf{k}) \otimes \tilde{H}^{q-|J|-1}(K_J;\mathbf{k}) \to \tilde{H}^{p+q-|I|-|J|-1}(K_{I\cup J};\mathbf{k})$$

that are induced by the canonical inclusion maps $\iota_{I,J} : K_{I\cup J} \to K_I * K_J$ *for* $I\cap J = \emptyset$ *and zero otherwise, where* $K_I * K_J$ *denotes the join of two simplicial complexes* K_I *and* K_J*.*

Here, we recall that if the moment-angle complex Z_K is homotopy equivalent to a suspension, then the multiplication and all higher Massey products in $H^*(Z_K;\mathbf{k})$ are trivial. For example, see Corollary 3.11 of [22]. That is, the following implication holds:

$$Z_K \text{ is homotopy equivalent to a suspension} \Longrightarrow\ K \text{ is Golod}, \tag{1.1}$$

where K is Golod if K is Golod over any field $\mathbf{k}$. This observation enables us to investigate the Golod property through the study of moment-angle complexes. One of the first studies in this direction was introduced by Grbić and Theriault [10]. They demonstrated that the moment-angle complex associated with a shifted simplicial complex is homotopy equivalent to a wedge of spheres. In [14], Kishimoto and the first author extended this result to dual sequentially Cohen–Macaulay complexes, and provided some new Golod complexes. In these studies, the following theorem concerning the decomposition of polyhedral products (see Definition 2.1), as introduced by Bahri, Bendersky, Cohen, and Gitler [2], plays an essential role.

Theorem 1.2 *([2]). Let* K *be a simplicial complex on* $[m]$ *and let* $(C\underline{X},\underline{X}) = \{(CX_i,X_i)\}_{i\in[m]}$*, where each* X_i *is a based space and* CX *is the reduced cone of a based space* X*. Then, the following homotopy equivalence holds:*

$$\Sigma Z_K(C\underline{X},\underline{X}) \simeq \Sigma \bigvee_{I\subset[m]} \Sigma|K_I| \wedge \widehat{X}^I,$$

where $\widehat{X}^I = \wedge_{i\in I} X_i$ *and* $\widehat{X}^\emptyset = *$*.*