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Continuous extension of functions from countable sets

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ABSTRACT

We give a characterization of countable discrete subspace A of a topological space X such that there exists a (linear) continuous mapping $\varphi : C_p^*(A) \to C_p(X)$ with $\varphi(y)|_A = y$ for every $y \in C_p^*(A)$. Using this characterization we answer two questions of A. Arhangel'skii. Moreover, we introduce the notion of well-covered subset of a topological space and prove that for well-covered functionally closed subset A of a topological space X there exists a linear continuous mapping $\varphi : C_p(A) \to C_p(X)$ with $\varphi(y)|_A = y$ for every $y \in C_p(A)$.

1. Introduction

For a topological space X we denote the space of all real-valued continuous function $y : X \to \mathbb{R}$ with the topology of pointwise convergence by $C_p(X)$, and the subspace of all real-valued continuous bounded function $y : X \to \mathbb{R}$ is denoted by $C_p^*(X)$.

According to the well-known Tietze–Urysohn theorem, for a normal space X and a closed subset A of X there exists a mapping $\varphi : C(A) \to C(X)$ such that $\varphi(y)|_A = y$ for every $y \in C(A)$. The existence and properties (linearity, continuity with respect different topologies, etc.) of such extender $\varphi : C(A) \to C(X)$ for various classes of spaces X were investigated by many mathematicians (see, for instance, [3,2,10–12,5,14,8] and literature given there). In particular, the existence of a linear continuous extender $\varphi : C_p(A) \to C_p(X)$ for every closed subset A of a stratifiable space X was obtained in [2, Theorem 4.3] and the existence of such extender for every closed subset A of locally compact generalized ordered space X was proved in [5, Corollary 1].

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The following questions were published in [1] (see also Questions 4.10.3 and 4.10.11 from [13]).

Question 1.1. Let X be a pseudocompact space such that for any countable set $A \subseteq X$ there exists a (linear) continuous map $\varphi : C_p^*(A) \to C_p(X)$ with $\varphi(y)|_A = y$ for every $y \in C_p(A)$. Must X be finite?

Question 1.2. Let X be the subspace of all weak P-points of $\beta\omega \setminus \omega$. Is it true that for any countable set $A \subseteq X$ there exists a (linear) continuous map $\varphi : C_p^*(A) \to C_p(X)$ such that $\varphi(y)|_A = y$ for every $y \in C_p(A)$?

A point x of a topological space X is called *an weak P-point* if $x \notin \overline{A}$ for every countable set $A \subseteq X \setminus \{x\}$.

In this paper we give a characterization of countable discrete subspace A of a topological space X for which there exists a (linear) continuous mapping $\varphi : C_p^*(A) \to C_p(X)$ with $\varphi(y)|_A = y$ for every $y \in C_p^*(A)$. Using this characterization we obtain the positive answer to Question 1.1 and the negative answer to Question 1.2. Moreover, we introduce the notion of well-covered subset of a topological space and prove that for well-covered functionally closed subset A of a topological space X there exists a linear continuous mapping $\varphi : C_p^*(A) \to C_p(X)$ with $\varphi(y)|_A = y$ for every $y \in C_p^*(A)$.

2. Countable *C*-embedding sets

The next property is probably well-known (see [10, Corollary 1]).

Proposition 2.1. *Let X be a completely regular space and $A \subseteq X$ such that there exists a continuous mapping $\varphi : C_p^*(A) \to C_p(X)$ such that $\varphi(y)|_A = y$ for every $y \in C_p^*(A)$. Then the set A is closed in X.*

Proof. Suppose that $x_0 \in \overline{A} \setminus A$. Let $y_0 \in C(A)$ such that $y_0(x) = 0$ for every $x \in A$ and $z_0 = \varphi(y_0)$. Clearly that $z_0(x_0) = 0$. Consider the neighborhood $W_0 = \{z \in C_p(X) : z(x_0) < \frac{1}{2}\}$ of z_0 in $C_p(X)$ and choose an finite set $B \subseteq A$ such that $\varphi(y) \in W_0$ for every $y \in C_p^*(A)$ with $y|_B = y_0|_B$. We choose an open in X set G such that $B \subseteq G \cap A$ and $x_0 \notin \overline{G}$. There exists a continuous function $y \in C_p^*(A)$ such that $y(x) = 0$ for every $x \in B$ and $y(x) = 1$ for every $x \in A \setminus G$. It is easy to see that $y|_B = y_0|_B$ and $\varphi(y) \notin W_0$, which implies a contradiction. □

Definition 2.2. A set A in a topological space X is strongly functionally discrete in X if there exists a discrete family $(G_a; a \in A)$ of functionally open sets $G_a \ni a$.

Proposition 2.3. *Let X be a topological space, $A = \{a_n : n \in \mathbb{N}\} \subseteq X$ be a countable set, Y be a Hausdorff compact space and $f : X \times Y \to \mathbb{R}$ be a separately continuous function such that the continuous mapping $\varphi : Y \to C_p(A)$, $\varphi(y)(a) = f(a, y)$, is a homeomorphic embedding and for every $n \in \mathbb{N}$ there exist $y_n', y_n'' \in Y$ with $|f(a_k, y_n') - f(a_k, y_n'')| \le \frac{1}{n+1}$ for all $k = 1, \dots, n$ and $|f(a_k, y_n') - f(a_k, y_n'')| \ge 1$ for all $k > n$. Then the set A is a strongly functionally discrete in X.*

Proof. Without loss of generality we can suppose that $Y \subseteq C_p(A)$. For every $n \in \mathbb{N}$ we put

$$U_n = \bigcap_{k=1}^{n-1} \{x \in X : |f(x, y_k') - f(x, y_k'')| > \tfrac{5}{6}\} \bigcap \{x \in X : |f(x, y_n') - f(x, y_n'')| < \tfrac{2}{3}\}.$$

Since f is continuous with respect to the first variable, U_n is functionally open set in X for every $n \in \mathbb{N}$. Moreover, for every $n \in \mathbb{N}$ we have

$$|f(a_n, y_n') - f(a_n, y_n'')| \le \tfrac{1}{n+1} < \tfrac{2}{3}$$