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Convex sections of rectangular sets and splitting of selections

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In this paper we provide some affirmative results and some counterexamples for a solution of the splitting problem for n multivalued mappings, $n > 2$.

1. Preliminaries

To an n-tuple of multivalued mappings $F_k : X \to Y_k$, $k = 1, 2, \dots, n$, and a singlevalued mapping $L : \oplus Y_k \to Y$ one can associate the *composite* multivalued mapping, say $L \circ (\oplus F_k) : X \to Y$, which associates to each $x \in X$ the set

$$\{y \in Y : y = L(y_1; ...; y_n),\ y_k \in F_k(x),\ k = 1, 2, \dots, n\}.$$

Definition 1.1. Let $f : X \to Y$ be a selection of the composite mapping $L \circ (\oplus F_k)$. An n-tuple $(f_1, \dots, f_n)$ is said to be a **splitting** of f if $f_k : Y_k \to Y$ is a selection of F_k, $k = 1, 2, \dots, n$, and $f(x) = L(f_1(x); ...; f_n(x))$, $x \in X$.

Recall that a single-valued mapping $f : X \to Y$ is said to be a *selection* of a multivalued mapping $F : X \to Y$ provided that $f(x) \in F(x)$, for every $x \in X$. Having in mind the celebrated convexvalued selection theorem of Michael [7], below we focus on the case of continuous singlevalued selections of convex- and closedvalued LSC mappings from paracompact domains to Banach spaces.

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As an example, let C_k, $k = 1, 2, \ldots, n$ be convex closed subsets of $\mathbb{R}^N$, let $L : \mathbb{R}^N \oplus \ldots \oplus \mathbb{R}^N \to \mathbb{R}^N$ be linear, $X = L(C_1, \ldots, C_n)$ be the result of Minkowsky pointwise vector operation in the space $\mathbb{R}^N$ and $F_k(\cdot) \equiv C_k$, $k = 1, 2, \ldots, n$. Clearly, $L \circ (\oplus F_k)(\cdot) \equiv X$ and the identity mapping $f = id|_X$ is a continuous selection of $L \circ (\oplus F_k)$. So, a splitting of $f = id|_X$ gives a singlevalued solution $(y_1, \ldots, y_n)$ of the classical linear equation

$$L(y_1, \ldots, y_n) = y, \qquad y_1 \in C_1, \ldots, y_n \in C_n$$

which continuously depends on the data y.

As pointed out in [2,3], the question of an existence of splitting is closely related to various tasks of set-valued analysis, e.g. parametrization of multivalued mappings [1,9], see also [5,6,8]. The notion of splitting was introduced in [10]. This notion first appeared in personal communication with Prof. U. Marconi, who asked about a representation of an ε-selection of $F_1 + F_2$ as the sum of ε-selections of F_i, $i = 1, 2$.

So, for fixed multivalued mappings $F_k : X \to Y_k, k = 1, 2, \ldots, n$, and for a linear map $L : \oplus Y_k \to Y$, the *splitting problem* is the question of finding suitable $(f_1, \ldots, f_n)$ for any selection $f : X \to Y$ of $L \circ (\oplus F_k)$, meanwhile for spaces $Y_1, \ldots, Y_n, Y$ the *splitting problem* means the splitting problem for arbitrary LSC mappings $F_k : X \to Y_k, k = 1, 2, \ldots, n$ of a paracompact domain X and linear mappings $L : \oplus Y_k \to Y$.

Formally, the splitting problem for mappings can be easily reduced to a selection problem. Namely, for a fixed multivalued mappings $F_k : X \to Y_k$, $k = 1, 2, \ldots, n$, and for a linear $L : \oplus Y_k \to Y$ the following are equivalent:

(a) $(f_1, \ldots, f_n)$ splits a selection f of $L \circ (\oplus F_k)$;
(b) $(f_1(x), \ldots, f_n(x)) \in L^{-1}(f(x)) \bigcap (\oplus F_k(x))$, $x \in X$.

Clearly, all sets $L^{-1}(f(x)) \bigcap (\oplus F_k(x))$, $x \in X$, are nonempty, convex and closed. So, modulo Michael's selection theorem, in order to deduce (a) from (b) one in fact, needs the lower semicontinuity of the associated mapping $x \mapsto L^{-1}(f(x)) \bigcap (\oplus F_k(x))$, $x \in X$. But the lower semicontinuity property is too unstable with respect to the pointwise intersection of multivalued mappings. This is the principal and significant obstacle for finding of a splitting.

Known positive results [2–4,8,10,11] for the case $n = 2$ as a rule exploit some linking properties of values $F_1(\cdot)$, $F_2(\cdot)$ and $KerL$ together. For example, splitting problem can be successfully resolved for constant mappings $F_1(\cdot) \equiv A \subset Y_1$, $F_2(\cdot) \equiv B \subset Y_2$ with finite-dimensional strictly convex A and B and with $KerL$ which is transversal to $Y_1 \times 0$ and $0 \times Y_2$ in $Y_1 \oplus Y_2$, ([11], Theorem 3.5). For a generalizations of this fact to the case of Hausdorff continuous strictly convex-valued mappings see ([2], Theorem 2.10) and uniformly (or, weakly) convex-valued mappings ([4], Example 4.1)

The positive answer to the splitting problem not for mappings (F_k, $k = 1, 2, \ldots, n$, L) but for *spaces* (Y_k, Y) is quite rare, maybe because there are too many universal quantifiers in its statement: $\forall X, \forall F_k, \forall L, \forall f, \exists \ldots$. Examples in Section 3 show that except the case $Y_1 = \ldots = Y_n = Y = \mathbb{R}$ the splitting problem, in general, admits negative solutions. In ([10] Theorem 3.1) it was proved that splitting is always possible for spaces $Y_1 = Y_2 = Y = \mathbb{R}$. Here we propose a generalization to the case of an arbitrary $n \in \mathbb{N}$, see Theorem 2.4 below.

Remark: It is natural to try to prove this theorem by induction with the base $n = 2$. Unfortunately, even the first reduction from $n = 3$ to $n = 2$ in general doesn't work because for $Y_1 = \mathbb{R}^2$, $Y_2 = Y = \mathbb{R}$ negative answers for splitting are possible, see Proposition 3.3 below. So the independent proof for the case of an arbitrary $n \in \mathbb{N}$ is based on a good "tomography" of convex *rectangular subsets* of $\mathbb{R}^n$, i.e. sets which coincide with the Cartesian product of their projections onto the coordinate lines.

Ending the introduction we recall that lower semicontinuity of a multivalued mapping $F : X \to Y$ between topological spaces X and Y means that for each pair of points $x \in X$ and $y \in F(x)$, and each