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Common fixed points and coincidences of mapping families on partially ordered sets

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New coincidence and common fixed point theorems are proved for families of multivalued mappings on partially ordered sets. The obtained results generalize some recent authors' fixed point theorems for one isotone multivalued mapping, and some recent coincidence theorems for two multivalued mappings one of which is isotone and the other one is orderly covering.

This paper is a continuation of the authors' investigation of fixed point and coincidence existence problems for mappings of partially ordered sets started in [1]. The formulation of the fixed point problem considered here goes back to the papers [2,3]. The coincidence problem statement develops the ideas of the papers [4,5] on coincidences of two set-valued mappings between ordered sets. The last two cited papers are in turn based on the papers [6,7] devoted to investigation of coincidence problem of two single-valued mappings of ordered sets. We introduce new concepts such as concordantly isotone family of multivalued mappings, concordantly chain isotone family of multivalued mappings, and a concept of family of n mappings concordantly orderly covering another family of n mappings ($n \geq 1$). New common fixed point theorems are proved for (an infinite) family of multivalued self-mappings of a partially ordered set, generalizing some results of [1]. New coincidence theorems for a collection of $n(n \geq 2)$ multivalued mappings between partially ordered sets, generalizing some results of [4], are obtained as well. A parallel is drawn between the introduced concepts and obtained results on the one hand, and similar concepts and results of [4,1] on the other hand. We support our results by examples. The main results of this paper were announced in [8] without proofs.

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Everywhere below symbol $\rightrightarrows$ stands for a multivalued mapping. In an ordered set $(X, \preceq)$ we use denotation $\mathcal{O}_X(x) = \{x' \in X | x' \preceq x\}$ for $x \in X$.

1. Common fixed point results

Let $(X, \preceq)$ be a partially ordered set, $A \neq \varnothing$, and $\mathcal{F} = \{F_\alpha\}_{\alpha \in A}$ be a family of multivalued self-mappings $F_\alpha : X \rightrightarrows X$, $\forall \alpha \in A$.

Definition 1. A set $\{y_\alpha\}_{\alpha \in A} \subseteq X$ is called **a set of $\mathcal{F}$-values** at point $x \in X$, if $y_\alpha \in F_\alpha(x)$, $\forall \alpha \in A$.

Definition 2. We say family $\mathcal{F}$ is **concordantly isotone** if for any $x \in X$, for any set $\{y_\alpha\}_{\alpha \in A}$ of $\mathcal{F}$-values at point x, and for any point $x' \in X$, $x' \prec x$, there exists a set $\{z_\alpha\}_{\alpha \in A}$ of $\mathcal{F}$-values at point x', with $z_\alpha \preceq y_\beta$, $\forall \alpha, \beta \in A$.

Let us note that if a family of mappings is concordantly isotone, every mapping of this family is isotone.

Definition 3. Let X_1, X_2 be subsets of X, such that $X_1 \subseteq X_2$, and families $f' = \{f'_\alpha\}_{\alpha \in A}$ and $f = \{f_\alpha\}_{\alpha \in A}$ of single-valued mappings be given, where $f'_\alpha : X_2 \to X$, $f_\alpha : X_1 \to X$, $\alpha \in A$. We say family f' is an **extension** of family f, and write $f'|_{X_1} = f$, if and only if $f'_\alpha|_{X_1} = f_\alpha$, $\forall \alpha \in A$.

Let us note that given two ordered sets $(X, \preceq)$, $(Y, \preceq)$, an extension $f' = \{f'_\alpha\}_{\alpha \in A}$ of family $f = \{f_\alpha\}_{\alpha \in A}$ can be defined quite similarly for the case when $f'_\alpha : X_2 \to Y$, $f_\alpha : X_1 \to Y$, $\forall \alpha \in A$.

The next definition is a generalization, for the case of a family of mappings, of the definition of a special one-valued selector of a multivalued mapping given in [1].

Definition 4. We say a special $\mathcal{F}$-selector is defined on a chain $S \subseteq X$ if there exists a family of one-valued mappings $f = \{f_\alpha\}_{\alpha \in A}, f_\alpha : S \to X$, $\forall \alpha \in A$, and the following conditions hold.

a) Set $\{f_\alpha(x)\}_{\alpha \in A}$ is a set of $\mathcal{F}$-values at point x, for any $x \in S$;
b) $x \succeq f_\alpha(x)$, $\forall \alpha \in A$, $\forall x \in S$;
c) $\forall u, v \in S$, $v \prec u \Longrightarrow v \preceq f_\alpha(u)$, $\forall \alpha \in A$.

Let us fix an element $x_0 \in X$ and **denote by $\mathcal{C}_1(x_0; \mathcal{F})$ the set** of all pairs of the form (S, f), where $S \subseteq \mathcal{O}_X(x_0)$ is a chain, and f is a special $\mathcal{F}$-selector defined on chain S.

The set of common fixed points of family $\mathcal{F}$ is traditionally denoted by $Comfix(\mathcal{F}) := \{x \in X | x \in \bigcap\limits_{\alpha \in A} F_\alpha(x)\}$.

Theorem 1. *Let $(X, \preceq)$ be a partially ordered set, $\mathcal{F} = \{F_\alpha\}_{\alpha \in A}$ be a set of multivalued self-mappings on X, and the following conditions hold.*

1) *Family $\mathcal{F}$ is concordantly isotone;*
2) *for some point $x_0 \in X$, there exists a set $\{y^0_\alpha\}_{\alpha \in A}$ of $\mathcal{F}$-values at point x_0, such that $x_0 \succeq y^0_\alpha$, $\forall \alpha \in A$;*
3) *for any pair $(S, f) \in \mathcal{C}_1(x_0; \mathcal{F})$, there exists a common lower bound $w \in X$ of chains $f_\alpha(S)$, $\forall \alpha \in A$, and there is a set $\{w_\alpha\}_{\alpha \in A}$ of $\mathcal{F}$-values at w, such that $w \succeq w_\alpha$, $\forall \alpha \in A$.*

Then $Comfix(\mathcal{F})$ is a nonempty set containing a minimal element.