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Properties related to star countability and star finiteness

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A B S T R A C T

Two star properties recently studied by Song and a generalization of countable compactness called weak star finiteness by Song and previously, 1-cl-starcompactness by Matveev and Ikenaga, are studied. We show that two of these properties coincide with feeble Lindelöfness and feeble compactness respectively in the class of spaces with a dense set of isolated points. Preservation of one of these properties under products by compact and sequentially compact spaces is also considered.

1. Introduction and notation

If X is a topological space and $\mathcal{U}$ is a family of subsets of X, then the *star* of a subset $A \subseteq X$ with respect to $\mathcal{U}$ is the set $\mathsf{St}(A,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}$.

Suppose that P is a topological property; a space X is said to be *star P* if whenever $\mathcal{U}$ is an open cover of X, there is a subspace $A \subseteq X$ with property P such that $X = \mathsf{St}(A,\mathcal{U})$. The space X is *weakly star P* (respectively, *almost star P*) if given any open cover $\mathcal{U}$ of X, there is a subspace $A \subseteq X$ with property P and such that $\mathrm{cl}(\mathsf{St}(A,\mathcal{U})) = X$ (respectively, $\bigcup\{\mathrm{cl}(\mathsf{St}(\{x\},\mathcal{U})) : x \in A\} = X$). The set A will be called a *star*

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kernel (respectively, *weak star kernel, almost star kernel*) of the cover $\mathcal{U}$. A space is *feebly compact* if every family of locally finite non-empty open sets is finite; in the class of Tychonoff spaces feeble compactness is equivalent to pseudocompactness (and here, pseudocompact spaces are always assumed to be Tychonoff).

The term "star P" was coined in [10] but many star properties, including those corresponding to "$P =$ finite" and "$P =$ countable" were introduced by Ikenaga (see for example [7]) and first studied by van Douwen et al. in [3]; generalizations were later introduced by many other authors. A survey of star properties with a comprehensive bibliography can be found in [9]. We note that the properties here called "star countable" and "star finite" have been studied previously under various other names; for instance, in the survey [9] they are called "star-Lindelöf" and "starcompact" respectively. The properties "weakly star finite" and "weakly star countable" are also mentioned in this survey on pages 11 and 89, under the names "1-cl-starcompact" (also "1-H-closed") and "1-cl-star-Lindelöf" respectively, where the first of these terms is attributed to Ikenaga [7]. However, beyond the trivial implications which appear in [9]:

$$\text{Countably compact} \Rightarrow \text{1-cl-starcompact} \Rightarrow \text{pseudocompact}$$

in the class of Tychonoff spaces and

$$\text{Lindelöf} \Rightarrow \text{1-cl-star-Lindelöf},$$

in the class of Hausdorff spaces, little attention seems to have been given to these properties until recently.

The terms "weakly star countable" and "almost star countable" were used by Y-K. Song in [14] and [15] respectively, where a number of results concerning these properties were obtained.

An interesting problem is the subtle relationship between the properties of being weakly star countable, that of being weakly star finite and feeble compactness. Clearly, the second of these properties implies the first and as we mention in Section 3, weakly star finite, weakly regular spaces are feebly compact. However, in [16], a pseudocompact, first countable, locally compact Hausdorff space which is not weakly star countable and hence not weakly star finite, is constructed in ZFC (an earlier example under CH appears as Example 2.3.2 in [3]). We note that in both these last cited papers (although not in [9]), the term "2-starcompact" is equivalent to pseudocompact in the class of Tychonoff spaces and weakly star finite implies "strongly 2-starcompact" (see Diagram 3 in [9], but the notation there is confusingly different). We address the problem of this relationship in Section 3, while Section 2 contains results on weakly and almost star countable spaces.

Although in the survey [9] no separation axioms are assumed unless explicitly stated, all spaces in this paper are assumed to be (at least) Hausdorff unless otherwise specified and all undefined terms can be found in [4].

2. Generalizations of the star countable property

In [9], Matveev defines a space to have the property $DC(\omega_1)$ if has a dense subspace D such that every uncountable subset of D has an accumulation point in X. The following result, cited in [9], seems to be part of the folk-lore of the subject, but for completeness we give a proof.

Theorem 2.1. *If a space X has a dense subspace D such that every uncountable subset of D has an accumulation point in X, then X is weakly star countable.*

Proof. Let $\mathcal{U}$ be an open cover of X and pick $x_0 \in D$. If $\mathrm{cl}(\mathsf{St}(\{x_0\},\mathcal{U})) \supseteq D$, then we are done; otherwise choose $x_1 \in D \setminus \mathrm{cl}(\mathsf{St}(\{x_0\},\mathcal{U}))$. Having constructed discrete sets $A_\alpha = \{x_\gamma : \gamma < \alpha\}$ for each $\alpha < \beta$, if $\mathrm{cl}(\mathsf{St}(A_\alpha,\mathcal{U})) \supseteq D$, then the construction ends. Otherwise pick $x_\alpha \in D \setminus \mathrm{cl}(\mathsf{St}(A_\alpha,\mathcal{U}))$; the set $A_{\alpha+1} = \{x_\beta : \beta \le \alpha\}$ is clearly discrete. If for all $\alpha \in \omega_1$ we have that $D \nsubseteq \mathrm{cl}(\mathsf{St}(A_\alpha,\mathcal{U}))$, then by the hypothesis, A_{ω_1}