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Rigidity of the second symmetric product of the pseudo-arc

Irving Calderón [a], Rodrigo Hernández-Gutiérrez [b], Alejandro Illanes [a,*]

[a] *Instituto de Matemáticas, Universidad Nacional Autónoma de México Circuito exterior, Cd. Universitaria, México D.F., 04510 Mexico*
[b] *Department of Mathematics and Statistics, University of North Carolina at Charlotte, Charlotte NC 28223, United States*

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A B S T R A C T

Let P denote the pseudo-arc and let $F_2(P) = \{\{p, q\} : p, q \in P\}$ denote the second symmetric product of P. The main result in this paper is the following: if $E : F_2(P) \to F_2(P)$ is an embedding, then there is an embedding $e : P \to P$ such that $E(\{p, q\}) = \{e(p), e(q)\}$. We obtain that the autohomeomorphisms of $F_2(P)$ are induced, P has rigid hyperspace $F_2(P)$, and the degree of homogeneity of $F_2(P)$ is exactly 3.

1. Introduction

A *continuum* is a nondegenerate compact connected metric space and a *mapping* means a continuous function. Given a continuum X, the *nth-symmetric product* is defined as the hyperspace $F_n(X) = \{A \subset X : 1 \le |A| \le n\}$, with the Vietoris topology [9].

The pseudo-arc is the simplest nondegenerate hereditarily indecomposable continuum. It can be characterized as the unique hereditarily indecomposable chainable continuum. For the history and an overview of the pseudo-arc, see Lewis' survey paper [11]. In this paper we will denote the pseudo-arc by P.

* Corresponding author.
E-mail addresses: irvingdanielc@ciencias.unam.mx (I. Calderón), rodrigo.hdz@gmail.com (R. Hernández-Gutiérrez), illanes@matem.unam.mx (A. Illanes).

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One of the most interesting and unexpected properties of the pseudo-arc is that it is homogeneous, as was proved by Bing [2]. Clearly, the square of the pseudo-arc, $P \times P$, is also homogeneous. However, in [1], it was proved that there is certain rigidity in $P \times P$. Namely, every autohomeomorphism of $P \times P$ is of one of the two forms $h = h_0 \times h_1$ or $i \circ (h_0 \times h_1)$, where $h_0, h_1 : P \to P$ are homeomorphisms and $i(p, q) = (q, p)$ for all $p, q \in P$. This result has been extended [3] to embeddings of $P \times P$ to itself in the natural way (see Theorem 3.1 below).

A mapping $G : F_n(X) \to F_n(X)$ is *induced by a mapping* $g : X \to X$ if for each $A \in F_n(X)$, $G(A) = g(A)$ (the image of A under g).

The continuum X has *rigid hyperspace* $F_n(X)$ if for each homeomorphism $G : F_n(X) \to F_n(X)$ we have $G(F_1(X)) = F_1(X)$. This notion was introduced in [4,5] and [6] and was used to study uniqueness of hyperspaces.

The *degree of homogeneity* of the continuum X is the number of orbits of the action of the group of homeomorphisms of X onto itself. In recent years, the degree of homogeneity has been widely studied. In particular, in [13], some continua X for which the degree of homogeneity of $F_2(X)$ is exactly 2 have been obtained. In [7], the second-named author and Verónica Martínez-de-la-Vega have determined the degree of homogeneity of symmetric products of some continua, including simple closed curves and manifolds.

In this paper we are mainly interested in determining the nature of autohomeomorphisms of $F_2(P)$. We prove that embeddings from $F_2(P)$ into itself are induced, so this property also holds for autohomeomorphisms of $F_2(P)$. We also show that P has rigid hyperspace $F_2(P)$, and that the degree of homogeneity of $F_2(P)$ is exactly 3.

2. Preliminaries

For a reference on continuum theory and hyperspaces, see [9] and [12], respectively. Let $\mathbb{N}$ denote the set of positive integers. If X is a space, the closure of a subset A of X will be denoted by $\mathrm{cl}_X(A)$. If X is a metric space with metric d, $p \in X$ and $r > 0$, let $B^d(p, r) = \{q \in X : d(p, q) < r\}$.

Given a continuum X, besides the hyperspace $F_n(X)$, we will use also the *hyperspace of subcontinua* of X which is defined by

$$C(X) = \{A \subset X : A \text{ is nonempty, closed and connected}\}.$$

Both hyperspaces, $F_n(X)$ and $C(X)$, are considered with the Vietoris topology.

Fix a hyperspace $\mathcal{K}(X)$ of some continuum X. Given $U_1, \ldots, U_m \subset X$, let

$$\langle U_1, \ldots, U_m \rangle = \{A \in \mathcal{K}(X) : A \subset U_1 \cup \ldots \cup U_m \text{ and } A \cap U_j \neq \emptyset \text{ for all } j \leq m\}.$$

Let us recall that $\langle U_1, \ldots, U_m \rangle$ is open in $\mathcal{K}(X)$ whenever $U_1, \ldots, U_m$ are open sets in X. Whenever d is a metric on X, there exists a metric defined on $\mathcal{K}(X)$ generating the Vietoris topology, called the Hausdorff metric [9, p. 9], and we will denote it by H_d. Both, the definition of the sets $\langle U_1, \ldots, U_m \rangle$ and the Hausdorff metric depend on which hyperspace $\mathcal{K}(X)$ represents, but this will usually not cause confusion.

We will need the following lemma whose proof is standard so we will not include it.

Lemma 2.1. *Let X be a continuum, $m \leq n$ positive integers, and let $U_1, \ldots, U_m$ be pairwise disjoint nonempty open subsets of X. Then every component of $\langle U_1, \ldots, U_m \rangle$ in $F_n(X)$ is of the form $\langle C_1, \ldots, C_m \rangle$, where C_i is a component of U_i for each $i \leq m$.*

In [9, Theorem 14.6] it is shown that if X is a continuum, $A, B \in C(X)$ and $A \subsetneq B$, then there is an order arc from A to B; namely, there is a continuous function $\alpha : [0, 1] \to C(X)$ such that $\alpha(0) = A$, $\alpha(1) = B$, and if $0 \leq s < t \leq 1$, then $\alpha(s) \subsetneq \alpha(t)$.