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A continuum without non-block points

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For any composant $E \subset \mathbb{H}^*$ and corresponding near-coherence-class $\mathscr{E} \subset \omega^*$ we prove the following are equivalent: (1) E properly contains a dense semicontinuum. (2) Each countable subset of E is contained in a dense proper semicontinuum of E. (3) Each countable subset of E is disjoint from some dense proper semicontinuum of E. (4) $\mathcal{E}$ has a minimal element in the finite-to-one weakly-increasing order of ultrafilters. (5) $\mathcal{E}$ has a Q-point. A consequence is that NCF is equivalent to $\mathbb{H}^*$ containing no proper dense semicontinuum and no non-block points. This gives an axiom-contingent answer to a question of the author. Thus every known continuum has either a proper dense semicontinuum at every point or at no points. We examine the structure of indecomposable continua for which this fails, and deduce they contain a maximum semicontinuum with dense interior.

1. Introduction

Non-block points are known to always exist in metric continua [8,13]. Moreover it follows from Theorem 5 of [4] that every point of a metric continuum is included in a dense proper semicontinuum. We call a point with this property a coastal point. A coastal continuum is one whose every point is coastal.

The author's investigation of whether non-metric continua are coastal began in [1]. The problem was reduced to looking at indecomposable continua. Specifically it was shown that every non-coastal continuum X admits a proper subcontinuum K such that the quotient space X/K obtained by treating K as a single point is indecomposable and fails to be coastal (as a corollary this proves separable continua are coastal).

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Since every indecomposable continuum with more than one composant is automatically coastal, the heart of the problem rests in those indecomposable (necessarily non-metric) continua with exactly one composant. We henceforth call these *Bellamy continua*, after David Bellamy who constructed the first example in ZFC [3]. There are very few examples known. The most well-studied candidate is the Stone–Čech remainder $\mathbb{H}^*$ of the half line. The composant number of $\mathbb{H}^*$ is axiom sensitive, but under the axiom Near Coherence of Filters (NCF) the composant number is exactly one [6]. In the first section of this paper, we show under NCF that $\mathbb{H}^*$ has neither coastal nor non-block points. Thus there consistently exists a non-coastal continuum.

It remains unresolved whether such a continuum can be exhibited without auxiliary axioms. The only other Bellamy continua of which the author is aware arise from an inverse-limit process [3,16,17]. The process in fact yields a continuum with exactly two composants – which are then combined by identifying a point of each. The nature of this construction ensures that what used to be a composant is still a dense proper semicontinuum, and so these examples are easily shown to be coastal.

Thus every known Bellamy continuum is either coastal at every point or at none. One might wonder whether these are the only options. This question is addressed in the paper's final section, where we show what pathology a partially-coastal Bellamy continuum must display.

2. Notation and terminology

By a *continuum* we mean a compact connected Hausdorff space. We do not presume metrisability. The interior and closure of a subspace B are denoted B° and B^- respectively. The continuum X is said to be *irreducible* between two points $a, b \in X$ if no proper subcontinuum of X contains the subset $\{a, b\}$.

The space A is called *continuumwise connected* if for every two points $a, b \in A$ there exists a continuum $K \subset A$ such that $\{a, b\} \subset K$. We also call a continuumwise connected space a *semicontinuum*. Every Hausdorff space is partitioned into maximal continuumwise connected subspaces. These are called the *continuum components*. When X is a continuum and $S \subset X$ a subset, we call S *thick* to mean it is proper and has nonvoid interior. The point $p \in X$ of a continuum is called a *weak cut point* if the subspace $(X - p)$ is not continuumwise connected. If $a, b \in X$ are in different continuum components of $(X - p)$ we say that p is *between* a and b and write $[a, p, b]$.

When X is a continuum the *composant* $\kappa(p)$ of the point $p \in X$ is the union of all proper subcontinua that include p. Another formulation is that $\kappa(p)$ is the set of points $q \in X$ for which X is not irreducible between p and q. For any points $x, p \in X$ we write $\kappa(x; p)$ for the continuum component of x in $(X - p)$. The point $x \in X$ is called *coastal* to mean that $\kappa(x; p)$ is dense for some $p \in X$. We call $p \in X$ a *non-block point* if $\kappa(x; p)$ is dense for some $x \in X$. From the definition, a continuum has a coastal point if and only if it has a non-block point, if and only if it contains a dense proper semicontinuum.

Throughout ω^* is the space of nonprincipal ultrafilters on the set $\omega = \{0, 1, 2, \ldots\}$ with topology generated by the sets $\widetilde{D} = \{\mathcal{D} \in \omega^* : D \in \mathcal{D}\}$ for all subsets $D \subset \omega$. Likewise $\mathbb{H}^*$ is the space of nonprincipal closed ultrafilters on $\mathbb{H} = \{x \in \mathbb{R} : x \geq 0\}$ with topology generated by the sets $\widetilde{U} = \{\mathcal{A} \in \mathbb{H}^* : A \subset U$ for some $A \in \mathcal{A}\}$ for all open subsets $U \subset \mathbb{H}$. For background on such spaces the reader is directed to [11] and [22].

$\mathbb{H}^*$ is known to be a *hereditarily unicoherent* continuum. That is to say any pair of its subcontinua have connected intersection. Moreover $\mathbb{H}^*$ is *indecomposable*, meaning we cannot write it as the union of two proper subcontinua. This is equivalent to every proper subcontinuum having void interior. The composants of an indecomposable continuum are pairwise disjoint.

For any two subsets $A, B \subset \mathbb{H}$ we write $A < B$ to mean $a < b$ for each $a \in A$ and $b \in B$. By a *simple sequence* we mean a sequence $I_n = [a_n, b_n]$ of closed intervals of $\mathbb{H}$ such that $I_1 < I_2 < I_3 < \ldots$ and the sequence a_n tends to infinity. Suppose $\mathbb{I} = \{I_1, I_2, \ldots\}$ is a simple sequence. For each subset $N \subset \omega$ define $I_N = \bigcup\{I_n : n \in N\}$. Then for each $\mathcal{D} \in \omega^*$ the set $\mathbb{I}_\mathcal{D} = \bigcap\{I_D^- : D \in \mathcal{D}\}$ is a subcontinuum of $\mathbb{H}^*$. These are called *standard subcontinua*. In case each sequence element is the singleton $\{a_n\}$ the corresponding standard subcontinuum is also a singleton, called a *standard point*, and we denote it by $a_\mathcal{D}$.