Topology and its Applications 219 (2017) 141–151

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

The region index and the unknotting number of a knot

Toshifumi Tanaka

Department of Mathematics, Faculty of Education, Gifu University, Yanagido 1-1, Gifu, 501-1193, Japan

ARTICLE INFO

Article history:
Received 7 December 2016
Received in revised form 23 January 2017
Accepted 25 January 2017
Available online 30 January 2017

MSC:
primary 57M25
secondary 57M27

Keywords:
Knots
Unknotting number
Region crossing change
Goeritz invariant
Goeritz matrix

ABSTRACT

For any knot in the 3-sphere S^3, there exists a diagram such that we have the unknot if we change all crossings on the boundary of some region of the diagram. The minimal number of the crossing changes over all such diagrams is called the region index of a knot. Clearly, the unknotting number is less than or equal to the region index. In this paper, we show that there exists a knot which has a gap between the unknotting number m and the region index for any positive integer m ($m \geq 2$) by using the Goeritz invariant. We also show that there exists a knot which has the unknotting number and the region index that are equal to n for any positive integer n ($n \geq 2$) by using the Rasmussen invariant.

1. Introduction

Let D be a diagram of a knot in S^3 and let $D^\times$ be the four-valent graph obtained from D by replacing each crossing with a vertex. We call (the closure of) each connected component of $S^2 - D^\times$ a *region* of D. A *region crossing change* at a region R of D is the local transformation on D by changing all the crossings on the boundary of R [11]. It is known that, for any knot K, there exists a diagram D such that D can be transformed into a diagram of the unknot by a single region crossing change at some region R of D [1]. We call such a diagram an *unknotting region diagram* and call the region R an *unknotting region* of D. We define the *region index* of a knot K, denoted by $\mathrm{Reg}(K)$, as the minimal number of crossings on the boundary of an unknotting region over all unknotting region diagrams of K. The region index is originally introduced by A. Kawauchi, K. Kishimoto and A. Shimizu in 2013. The *unknotting number* of a knot K, denoted by $u(K)$, is the minimal number of crossing changes needed to create the unknot, the minimum being taken over all possible sets of crossing changes in all diagrams of K [7,8]. Clearly, the unknotting number is less than or equal to the region index.

E-mail address: tanakat@gifu-u.ac.jp.

http://dx.doi.org/10.1016/j.topol.2017.01.012

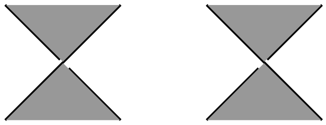

Fig. 1. The index $\zeta(c)$ at a crossing c.

In this paper, we consider the Goeritz matrix which is constructed from a diagram of a knot as follows. Let D be a diagram of a knot K in S^3. We consider a checkerboard coloring to the regions of D and allocate an index $\zeta(c)$, as in Fig. 1, to each crossing c of D.

Denote by $R_0, \ldots, R_n$ the black regions. The *pre-Goeritz matrix* of D is defined to be the $(n+1) \times (n+1)$-matrix with entries $g_{i,j} = \sum \zeta(c)$ $(i \neq j)$, the sum is over all crossings where R_i meets R_j and $g_{i,i} = -\sum_{j \neq i} g_{i,j}$ (i.e. the diagonal entries are chosen so that the sum of all entries of each row is zero). The matrix G_D obtained from the pre-Goeritz matrix by deleting one row and one column is called a *Goeritz matrix* of D [8].

The effects on G_D by the Reidemeister moves on D are given as follows [2,12]:

(a) $G_D \leftrightarrow VG_DV^T$, where V is a unimodular matrix, or
(b) $G_D \leftrightarrow G_D \oplus (\pm 1)$.

We define an equivalence relation on the set of integral square matrices by the following operations:

(1) $A \sim VAW$, where V and W are unimodular matrices, or
(2) $A \sim A \oplus (\pm 1)$.

Then two integral square matrices related by a finite sequence of the following operations:

(i) interchanging two rows (or two columns),
(ii) multiplying a row (or column) by -1, and
(iii) adding to any row a $\mathbb{Z}$-linear combination of other rows (or to any column a $\mathbb{Z}$-linear combination of other columns)

are equivalent. We call these operations (i), (ii) and (iii) *row operations* (or *column operations*).

The elementary divisor theory implies that G_D can be transformed uniquely, by performing a finite number of these operations, to a diagonal matrix with the non-negative diagonal entries $k_1, \ldots, k_s$ such that if $s \geq 2$ then k_i is an integer with $k_i \neq 1$ for $1 \leq i \leq s$ and divides k_{i+1} for $1 \leq i < s$. (See [5] for example.) The sequence $k(K) = (k_1, \ldots, k_s)$ of length s is called the *Goeritz invariant* of K [3,4]. We denote the length s by $d(K)$. Moreover, $|\det G_D| = k_1, \ldots, k_s$ is called the *determinant* of K (see [8]). Clearly, $\det K = 1$ if K is the unknot. The Goeritz invariants for pretzel links, two-bridge links and torus links are investigated in [3] and [4].

We can easily see that there is a gap between the region index and the unknotting number for any knot with unknotting number one. In this paper, we obtain the following results.

Theorem 1.1. *For any positive integer n with $n \geq 3$, there exists a knot K in S^3 such that $u(K) + 1 = \mathrm{Reg}(K) = n$.*

Theorem 1.2. *For any positive integer n with $n \geq 2$, there exists a knot K in S^3 such that $\mathrm{Reg}(K) = u(K) = n$.*