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Geometric intersection in representations of mapping class groups of surfaces

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ABSTRACT

We show that the detection of geometric intersection in an arbitrary representation of the mapping class group of surface implies the injectivity of that representation up to center, and vice versa. As an application, we discuss the geometric intersection in the Johnson filtration. Also, we further consider the problem of detecting the geometric intersection between separating simple closed curves in a representation.

1. Introduction

Detecting geometric intersection can be a powerful tool for the study of representations of the mapping class group of surface. For instance, a certain kind of such detection in the Lawrence–Krammer representation by Bigelow [2] led to an affirmative solution to the linearity problem for Artin's braid group which is nothing but the mapping class group of a punctured disk. Conversely, the impossibility of detecting a similar kind of geometric intersection had led to the unfaithfulness results for the Burau representation of the braid group as shown by Moody [12], Long–Paton [11], and Bigelow [1]. As for the mapping class group of a surface of higher genus, this type of result was given by Suzuki [16] for the Magnus representation of the Torelli group. In each of all these works, it was fundamental to establish a criterion that the representation in question can detect the geometric intersection if and/or only if its kernel is small.

In this paper, instead of considering any particular representation, we derive a similar criterion applicable to an *arbitrary group homomorphism* of the mapping class group of a surface of genus at least one, by focusing our attention on the following fact:

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The geometric intersection number between two simple closed curves is zero if and only if the commutator of the two Dehn twists along them represents the identity in the mapping class group.

We now describe our main result. Let $\Sigma_{g,n}$ be an oriented compact connected surface of genus $g \geq 1$ with $n \geq 0$ boundary components. The mapping class group $\mathcal{M}_{g,n}$ of $\Sigma_{g,n}$ is defined as the group of all the isotopy classes of orientation preserving homeomorphisms of $\Sigma_{g,n}$ where all homeomorphisms and isotopies are assumed to preserve the boundary of $\Sigma_{g,n}$ pointwise. Let $\mathcal{S}$ be the set of all the isotopy classes of essential simple closed curves on $\Sigma_{g,n}$. Here, essential is meant to be not homotopic to a point nor parallel to any of the boundary components. For $c \in \mathcal{S}$, we denote by t_c the right-handed Dehn twist along c. We denote by $\mathcal{S}^{\text{nonsep}}$ the subset of $\mathcal{S}$ consisting of all the isotopy classes of nonseparating simple closed curves. The *commutator* of two elements x and y in a group is defined by $[x, y] = xyx^{-1}y^{-1}$.

Our criterion states that the triviality of geometric intersection number for *all* pairs of essential simple closed curves can be detected by a homomorphisms of $\mathcal{M}_{g,n}$ if and only if its kernel is small:

Theorem 1.1. *Let G be a group and $\rho : \mathcal{M}_{g,n} \to G$ an arbitrary homomorphism. If $[t_{c_1}, t_{c_2}] = 1$ for those c_1, $c_2 \in \mathcal{S}^{nonsep}$ which satisfy $\rho([t_{c_1}, t_{c_2}]) = 1$, then the kernel of ρ is contained in the center $Z(\mathcal{M}_{g,n})$ of $\mathcal{M}_{g,n}$. Conversely, if $\operatorname{Ker}\rho \subset Z(\mathcal{M}_{g,n})$, then $[t_{c_1}, t_{c_2}] = 1$ for any c_1 and $c_2 \in \mathcal{S}$ which satisfy $\rho([t_{c_1}, t_{c_2}]) = 1$.*

Note that the curves c_1 and c_2 need not be nonseparating in the latter half of Theorem 1.1.

Remark 1.2. The structure of the center $Z(\mathcal{M}_{g,n})$ is well-known due to Paris–Rolfsen [14]. If $n = 0$, $Z(\mathcal{M}_{g,n})$ is trivial except for the case $g \leq 2$, where the center is generated by the class of hyperelliptic involution. For the case of $g = 1$ and $n = 1$, the center is an infinite cyclic group generated by the "half-twist" along the unique boundary component. For all the other cases, $Z(\mathcal{M}_{g,n})$ is a free abelian group of rank n and is generated by the Dehn twists along the boundary components of $\Sigma_{g,n}$.

The organization of this paper is as follows. The proof of Theorem 1.1 is given in Section 2 after necessary preparation. In Section 3, as an application, we discuss the geometric intersection in the Johnson filtration and pose a certain problem. Also, in Section 4, we further consider the geometric intersection between *separating* simple closed curves and provide a criterion similar to Theorem 1.1. In Appendix A, we give a proof of certain key lemma for Section 4.

2. Proof of Theorem 1.1

We first prepare some necessary results. We refer to [4] as basic reference for mapping class groups of surfaces. We also need some results in our previous work [9] with certain modification.

For a, $b \in \mathcal{S}$, the geometric intersection number, denoted by $i_{\text{geom}}(a, b)$, is the minimum of the number of the intersection points of the simple closed curves α and β where α and β vary the isotopy classes of a and b, respectively. It defines a function

$$i_{\text{geom}} : \mathcal{S} \times \mathcal{S} \to \mathbb{Z}_{\geq 0}.$$

The following is the precise statement of the fact mentioned in Introduction (cf. Fact 3.9 in [4]).

Lemma 2.1. *For c_1, $c_2 \in \mathcal{S}$, $i_{geom}(c_1, c_2) = 0$ if and only if $[t_{c_1}, t_{c_2}] = 1$ in $\mathcal{M}_{g,n}$.*

The next is also well-known, and will be crucial in our argument.

Lemma 2.2. *Suppose c_1, $c_2 \in \mathcal{S}$. If $c_1 \neq c_2$, then there exists $d \in \mathcal{S}$ such that $i_{geom}(c_1, d) = 0$ and $i_{geom}(c_2, d) \neq 0$.*