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Nullifying randomness and genericity using symmetric difference

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ABSTRACT

For a class $\mathcal{C}$ of sets, let us say that a set A is $\mathcal{C}$ *stabilising* if $A \triangle X \in \mathcal{C}$ for every $X \in \mathcal{C}$. We prove that the Martin–Löf stabilising sets are exactly the K-trivial sets, as are the weakly 2-random stabilising sets. We also show that the 1-generic stabilising sets are exactly the computable sets.

1. Introduction

Lowness for randomness is a topic that has attracted much attention in the literature. Recall that a set A is *low for Martin–Löf randomness* if, whenever X is Martin–Löf random, X is also Martin–Löf random relative to A. Thus, if A is not low for Martin–Löf randomness, there is a Martin–Löf random set X such that A "derandomises" it.

One particular, very simple way in which an oracle A might derandomise a set X is if the symmetric difference $X \triangle A$ of X with A is not itself random. Here $X \triangle A = (X \setminus A) \cup (A \setminus X)$; or equivalently, if we identify sets with their indicator function, symmetric difference is the same as bitwise addition modulo 2. At first sight, it might seem that this method of derandomising a set is too weak to capture exactly those A that are not low for Martin–Löf randomness: it is very uniform, and also very local. Furthermore, it is a

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priori not even clear that the class of such A is degree-invariant, or that it is countable. However, at least the locality does not have to be a problem, since Nies [9] has shown that the (global) property of being low for Martin–Löf randomness corresponds to the (much more localised) property of being K-trivial.

Note that 2^ω forms an (abelian) group under the operation $\triangle$. In particular, we can view $\triangle$ as a group action $2^\omega \times 2^\omega \to 2^\omega$. Recall that, for any group action $G \times X \to X$, the *set stabiliser* of a subset $Y \subseteq X$ is the set

$$\{g \in G \mid gY = Y\}.$$

In case G is a torsion group (i.e., all elements have finite order), like 2^ω, note that

$$\{g \in G \mid gY = Y\} = \{g \in G \mid gY \subseteq Y\}.$$

Following this terminology, let us therefore make the following definition.

Definition 1.1. Let $\mathcal{C} \subseteq 2^\omega$. The *stabiliser* of $\mathcal{C}$ is the set

$$\{A \in 2^\omega \mid \forall X \in \mathcal{C}(A \triangle X \in \mathcal{C})\}.$$

We will say that such an A is $\mathcal{C}$ *stabilising*.

The question of which sets are Martin–Löf stabilising has circulated in the effective randomness community (using various terminology). Kjos-Hanssen was probably the first person to ask it. The first time the question appeared in print seems to be in Kihara and Miyabe [4]. They study the stabiliser of various notions from randomness and genericity, mainly for its connection to the cardinal characteristic of null-additivity.

Kihara and Miyabe gave characterisations of the $\mathcal{C}$ stabiliser for several classes $\mathcal{C}$. Recall that, for a randomness notion $\mathcal{R}$, a set A is *uniformly low for $\mathcal{R}$-randomness* if every $\mathcal{R}$-random set X passes $\mathcal{T}^A$ for all $\mathcal{T}$ such that $\mathcal{T}^B$ is an $\mathcal{R}$-test for *every* oracle B (Miyabe [7] and Miyabe and Rute [8]). Recall that, for Martin–Löf randomness and 1-genericity, uniform lowness and lowness coincide. On the other hand, for many other notions this is not the case. Furthermore, as noted above, the map $X \mapsto A \triangle X$ is uniform in A, and therefore one would expect that a connection between $\mathcal{R}$ stabilising and lowness for $\mathcal{R}$-randomness would, if it exists at all, refer to uniform lowness instead of non-uniform lowness.

It turns out such a connection often exists. In [4], it is shown that the Kurtz stabilising sets are exactly the sets that are uniformly low for Kurtz randomness, and that the weakly 1-generic stabilising sets are those that are uniformly low for weak 1-genericity. Furthermore, if we generalise the notion of $\mathcal{C}$ stabilising to the notion of $(\mathcal{C}, \mathcal{D})$ stabilising, where A is $(\mathcal{C}, \mathcal{D})$ *stabilising* if for every $X \in \mathcal{C}$ we have that $A \triangle X \in \mathcal{D}$, then they have shown that (Martin–Löf, Schnorr) stabilising coincides with uniformly (Martin–Löf, Schnorr) low, and that (Martin–Löf, Kurtz) stabilising coincides with uniformly (Martin–Löf, Kurtz) low. Here, recall that A is $(\mathcal{C}, \mathcal{D})$ *low* if every $\mathcal{C}$-random is $\mathcal{D}$-random relative to A, and it is straightforward to formulate the uniform version of this.

We add several results to this list. We show that the Martin–Löf stabilising sets are those that are low for Martin–Löf randomness (i.e., the K-trivial sets), and that the 1-generic stabilising sets are exactly the sets that are low for 1-generic (i.e., computable, by Greenberg, Miller and Yu, as published in Yu [11]). We also show that the weakly 2-random stabilising sets are the sets that are low for weak 2-randomness, and that the sets that are (weakly 2-random, Martin–Löf) stabilising are those that are (weakly 2-random, Martin–Löf) low. Note that these last two lowness classes also coincide with the K-trivial sets.

As noted above, these new characterisations of K-triviality are not obviously degree invariant. This is somewhat unusual. Of the seemingly countless characterisations of K-triviality that have been found, almost all of them say that K-trivial sets are weak as oracles, or that they are easy to compute (or both), in other

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