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Mitchell's theorem revisited

Thomas Gilton [a], John Krueger [b,*]

[a] *Department of Mathematics, University of California, Los Angeles, Box 951555, Los Angeles, CA 90095-1555, United States*
[b] *Department of Mathematics, University of North Texas, 1155 Union Circle #311430, Denton, TX 76203, United States*

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ABSTRACT

Mitchell's theorem on the approachability ideal states that it is consistent relative to a greatly Mahlo cardinal that there is no stationary subset of $\omega_2 \cap \mathrm{cof}(\omega_1)$ in the approachability ideal $I[\omega_2]$. In this paper we give a new proof of Mitchell's theorem, deriving it from an abstract framework of side condition methods.

Contents

* Corresponding author.
E-mail addresses: tdgilton@math.ucla.edu (T. Gilton), jkrueger@unt.edu (J. Krueger).

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Introduction

The approachability ideal $I[\lambda^+]$, for an uncountable cardinal λ, is defined as follows. For a given sequence $\vec{a} = \langle a_i : i < \lambda^+ \rangle$ of bounded subsets of λ^+, let $S_{\vec{a}}$ denote the set of limit ordinals $\alpha < \lambda^+$ for which there exists a set $c \subseteq \alpha$, which is club in α with order type $\mathrm{cf}(\alpha)$, such that for all $\beta < \alpha$, there is $i < \alpha$ with $c \cap \beta = a_i$. Intuitively speaking, the set $S_{\vec{a}}$ carries a kind of weak square sequence, namely a sequence of clubs such that for each α in $S_{\vec{a}}$, the club attached to α has its initial segments enumerated at stages prior to α. Define $I[\lambda^+]$ as the collection of sets $S \subseteq \lambda^+$ for which there exists a sequence $\vec{a}$ as above and a club $C \subseteq \lambda^+$ such that $S \cap C \subseteq S_{\vec{a}}$. In other words, $I[\lambda^+]$ is the ideal of subsets of λ^+ which is generated modulo the club filter by sets of the form $S_{\vec{a}}$.

Let λ be a regular uncountable cardinal. Shelah [14] proved that the set $\lambda^+ \cap \mathrm{cof}(< \lambda)$ is in $I[\lambda^+]$. Therefore the structure of $I[\lambda^+]$ is determined by which subsets of $\lambda^+ \cap \mathrm{cof}(\lambda)$ belong to it. At one extreme, the weak square principle $\square^*_\lambda$ implies that $\lambda^+ \cap \mathrm{cof}(\lambda)$ is in $I[\lambda^+]$; therefore $I[\lambda^+]$ is just the power set of λ^+. The opposite extreme would be that no stationary subset of $\lambda^+ \cap \mathrm{cof}(\lambda)$ belongs to $I[\lambda^+]$, in other words, that $I[\lambda^+]$ is the nonstationary ideal when restricted to cofinality λ. Whether the second extreme is consistent was open for several decades, and was eventually solved by Mitchell [12]. Mitchell proved that it is consistent, relative to the consistency of a greatly Mahlo cardinal, that there does not exist a stationary subset of $\omega_2 \cap \mathrm{cof}(\omega_1)$ in $I[\omega_2]$. We will refer to this result as *Mitchell's theorem*.

Mitchell's theorem is important not only for solving a deep and long-standing open problem in combinatorial set theory, but also for introducing powerful new techniques in forcing. A basic tool in the proof is a forcing poset for adding a club subset of ω_2 with finite conditions, using finite sets of countable models as side conditions. A similar forcing poset was introduced by Friedman [3] around the same time. The use of countable models in Friedman's and Mitchell's forcing posets for adding a club expanded the original side condition method of Todorčević [15], which was designed to add a generic object of size ω_1, to adding a generic object of size ω_2. In addition, Mitchell's proof introduced the new concepts of strongly generic conditions and strongly proper forcing posets, which are closely related to the approximation property.

Several years later, Neeman [13] developed a general framework of side conditions, which he called *sequences of models of two types*. An important distinction between Neeman's side conditions and those of Friedman and Mitchell is that the two-type side conditions include both countable and uncountable models. A couple of years later, Krueger [9] developed an alternative framework of side conditions called *adequate sets*. This approach bases the analysis of side conditions on the ideas of the *comparison point* and *remainder points* of two countable models. Notably, this approach has led to the solution of an open problem of Friedman [3], by showing how to add a club subset of ω_2 with finite conditions while preserving the continuum hypothesis ([10]). Other applications are given in [7,6,8], and [2].

Notwithstanding the merits of the frameworks of Neeman [13] and Krueger [9], these frameworks are limited in the sense that they are intended to add a single subset of ω_2 (or of a cardinal κ which is collapsed to become ω_2). The proof of Mitchell's theorem, on the other hand, involves adding κ^+ many club subsets of a cardinal κ. Many consistency proofs in set theory about a cardinal κ involve adding κ^+ many subsets of κ by forcing, so that each of the potential counterexamples to the statement being forced is captured in some intermediate generic extension and dealt with by the rest of the forcing extension.

The goal of this paper is to extend the framework of adequate sets to allow for adding many subsets of ω_2, or of a cardinal κ which is collapsed to become ω_2. The purpose of this extension is to provide general tools

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