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Honest elementary degrees and degrees of relative provability without the cupping property

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ABSTRACT

An element a of a lattice *cups* to an element $b > a$ if there is a $c < b$ such that $a \cup c = b$. An element of a lattice has the *cupping property* if it cups to every element above it. We prove that there are non-zero honest elementary degrees that do not have the cupping property, which answers a question of Kristiansen, Schlage-Puchta, and Weiermann. In fact, we show that if $\mathbf{b}$ is a sufficiently large honest elementary degree, then $\mathbf{b}$ has the *anti-cupping property*, which means that there is an $\mathbf{a}$ with $\mathbf{0} <_{\mathrm{E}} \mathbf{a} <_{\mathrm{E}} \mathbf{b}$ that does not cup to $\mathbf{b}$. For comparison, we also modify a result of Cai to show, in several versions of the degrees of relative provability that are closely related to the honest elementary degrees, that in fact all non-zero degrees have the anti-cupping property, not just sufficiently large degrees.

1. Introduction

An element a of a lattice *cups* to an element $b > a$ if there is a $c < b$ such that $a \cup c = b$. An element a of a lattice has the *cupping property* if it cups to every $b > a$. An element b of a lattice with 0 has the *anti-cupping property* if there is an a with $0 < a < b$ that does not cup to b. So, if $a, b > 0$ in a lattice with 0, a witnesses that b has the anti-cupping property if and only if b witnesses that a does not have the cupping property. In this work, we study cupping in several related lattices arising from elementary functions and total algorithms.

The first lattice we consider is the lattice $\mathcal{H}$ of *honest elementary degrees*, which arose from attempts to classify various sub-recursive classes of functions into hierarchies. In $\mathcal{H}$, the objects are (equivalence classes of) functions whose graphs are elementary relations, and these functions are compared via the 'elementary in' relation. The basic theory of this structure was developed by Meyer and Ritchie [21] and by Machtey [18–20].

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In recent years, intense work mainly by Kristiansen [11–17] has significantly advanced the theory. We refer the reader to [16] (and to the related [17]) for a survey of the area. In [17], the authors ask if every non-zero $\mathbf{a} \in \mathcal{H}$ has the cupping property. We answer this question negatively by showing that if $\mathbf{b} \in \mathcal{H}$ is sufficiently large (in the sense of Definition 3.1), then $\mathbf{b}$ has the anti-cupping property (Corollary 3.4). Thus if $\mathbf{b} \in \mathcal{H}$ is sufficiently large, then every $\mathbf{a} \in \mathcal{H}$ witnessing that $\mathbf{b}$ has the anti-cupping property is a non-zero degree that does not have the cupping property.

Next we consider two related families of lattices: the *degrees of provability* relative to arithmetical theories extending $\mathrm{I}\Sigma_1$ and the *honest α-elementary degrees* for ordinals $\alpha \leq \epsilon_0$ of the form ω^β. Let T be a consistent first-order theory in the language of arithmetic. In $\mathcal{P}_T$, the degrees of provability relative to T, the objects are (equivalence classes of) total algorithms (i.e., indices of total Turing machines), and these algorithms are compared via the 'provably total' relation. That is, $\deg_T(\Phi) \geq_T \deg_T(\Psi)$ if $T \vdash \mathrm{tot}(\Phi) \rightarrow \mathrm{tot}(\Psi)$, where $\mathrm{tot}(\Phi)$ is the sentence expressing the totality of the Turing machine Φ. Cai [7] introduced the degrees of relative provability in order to analyze the provability strengths of true Π_2 sentences or, equivalently, sentences expressing the totality of total algorithms. This line of research continues impressively in [2,8,9].

In $\mathcal{H}_\alpha$, the honest α-elementary degrees, the objects are again (equivalence classes of) functions whose graphs are elementary, and these functions are compared via the 'α-elementary in' relation, which coarsens the 'elementary in' relation by allowing functions to be iterated $\beta < \alpha$ many times. Kristiansen, Schlage-Puchta, and Weiermann [17] introduced the honest α-elementary degrees and proved a correspondence between the 'α-elementary in' relation and the 'provably total' relation relative to Peano arithmetic (PA). Such correspondences between sub-recursive hierarchies and provably total functions can be useful for analyzing the logical strengths of formal systems. See, for example, Beklemishev's work in [3–6].

The degrees of relative provability and the honest α-elementary degrees are very closely related. For a theory T, let T^+ be the extension of T by all true Π_1 sentences. Kristiansen [15] proves that $\mathcal{P}_{\mathrm{PA}^+}$ and $\mathcal{H}_{\epsilon_0}$ are isomorphic, and analogous results should hold for various fragments of PA and the appropriate ordinals.

Cai [9] proves that there are non-zero elements of $\mathcal{P}_{\mathrm{PA}^+}$ that do not have the cupping property. It follows from Kristiansen's isomorphism that there are also non-zero elements of $\mathcal{H}_{\epsilon_0}$ that do not have the cupping property. We modify Cai's result to prove that if T is a consistent, recursively axiomatizable theory extending $\mathrm{I}\Sigma_1$, then every non-zero $\mathbf{b} \in \mathcal{P}_{T^+}$ has the anti-cupping property (Corollary 5.3). Consider then the following two statements:

($\star$) Every b that is sufficiently large (where the meaning of 'sufficiently large' depends on the lattice in question) has the anti-cupping property.

(†) Every $b > 0$ has the anti-cupping property.

Corollary 3.4 states that ($\star$) holds in $\mathcal{H}$. By modifying the argument, we also see that ($\star$) holds in the $\mathcal{H}_\alpha$'s. Corollary 5.3 states that (†) holds in $\mathcal{P}_{T^+}$ for every consistent, recursively axiomatizable theory T extending $\mathrm{I}\Sigma_1$. In particular, (†) holds in $\mathcal{P}_{\mathrm{PA}^+}$ and so, by Kristiansen's isomorphism, also in $\mathcal{H}_{\epsilon_0}$. Thus the natural question is whether or not (†) holds in $\mathcal{H}$ and in every $\mathcal{H}_\alpha$. We expect that (†) holds in many of the $\mathcal{H}_\alpha$'s by extending Kristiansen's isomorphism result to fragments of PA.

2. Honest elementary degrees

In this section, we provide a basic introduction to the theory of the honest elementary degrees. Again, we refer the reader to [16,17] for more comprehensive surveys.

Definition 2.1.

- The *elementary functions* are those functions $f : \omega^n \rightarrow \omega$ that can be generated from the *initial elementary functions* by the *elementary definition schemes*.

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