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Ceres in intuitionistic logic

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In this paper we present a procedure allowing the extension of a CERES-based cut-elimination method to intuitionistic logic. Previous results concerning this problem manage to capture fragments of intuitionistic logic, but in many essential cases structural constraints were violated during normal form construction resulting in a classical proof. The heart of the CERES method is the resolution calculus, which ignores the structural constraints of the well known intuitionistic sequent calculi. We propose, as a method of avoiding the structural violations, the generalization of resolution from the resolving of clauses to the resolving of cut-free proofs, in other words, what we call *proof resolution*. The result of proof resolution is a cut-free proof rather than a clause. Note that resolution on ground clauses is essentially atomic cut, thus using proof resolution to construct cut-free proofs one would need to *join* the two proofs together and remove the atoms which where resolved. To efficiently perform this joining (and guarantee that the resulting cut-free proof is intuitionistic) we develop the concept of *proof subsumption* (similar to clause subsumption) and *indexed resolution*, a refinement indexing atoms by their corresponding positions in the cut formula. Proof subsumption serves as a tool to prove the completeness of the new method CERES-i, and indexed resolution provides an efficient strategy for the joining of two proofs which is in general a nondeterministic search. Such a refinement is essential for any attempt to implement this method. Finally we compare the complexity of CERES-i with that of Gentzen-based methods.

1. Introduction

Cut-elimination was originally introduced by Gerhard Gentzen as a theoretical tool from which results like decidability and consistency could be proven [10]. Cut-free proofs are computationally explicit objects from which interesting information such as Herbrand disjunctions and interpolants can easily be extracted.

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When viewing formal proofs as a model for mathematical proofs, cut-elimination corresponds to the removal of lemmas, which leads to interesting applications (see, e.g. [2,3]).

For such applications to mathematical proofs, the cut-elimination method CERES (cut-elimination by resolution) was developed in [5]; initially, the method was developed for classical first-order logic. It essentially reduces cut-elimination for a proof φ of a sequent S to a theorem proving problem: the refutation of a clause set corresponding to φ, denoted by $\mathrm{CL}(\varphi)$. Given a resolution refutation of $\mathrm{CL}(\varphi)$, an essentially cut-free proof (a proof with only atomic cuts) can be constructed by a proof-theoretic transformation. This proof theoretic transformation uses so-called proof projections $\varphi[C]$ for $C \in \mathrm{CL}(\varphi)$, which are simple cut-free proofs extracted from φ (proving the end-sequent S extended by the atomic sequent C).

Due to the reduction to a theorem proving problem encoding crucial structural properties of cut, CERES turned out to be a powerful tool in proof analysis [3]. Moreover, its asymptotic complexity is superior to that of two reductive Gentzen-style methods [7]. The original method CERES was designed for classical first-order logic. Extensions to higher-order logic and first-order proof schemata were defined in [12] and [9], respectively. As intuitionistic proofs, like classical ones, are natural formalisms for mathematical reasoning, they are of major importance to proof mining (see e.g. [13]). Therefore, it is a natural question whether the method CERES can be extended to intuitionistic logic. However, the naive extension of CERES to first-order intuitionistic logic does not work, as the results of the CERES-transformations are classical proofs in general. In [16], it was shown that, for some intuitionistic proofs, there are refutations of the clause set which cannot be transformed into intuitionistic proofs: neither are the CERES-normal forms intuitionistic nor can they be transformed into intuitionistic proofs by reasonable proof transformations. Only for proofs φ of sequents of the form $\Gamma \vdash$, the CERES-method can be maintained, provided negative resolution refutations are applied to $\mathrm{CL}(\varphi)$ [16]. This suggests that, in order to cover all intuitionistic proofs, a radical change of the CERES-method is required.

In this paper, we develop a complete CERES-like method for intuitionistic proofs φ with skolemized end-sequents, called CERES-i. In Section 3.1, we show that a separation of projections and resolution refutations (which is characteristic to first-order CERES) does not work for intuitionistic logic; in fact there are proofs φ of a sequent S and resolution refutations of $\mathrm{CL}(\varphi)$ which cannot be combined with the projections to construct an intuitionistic cut-free proof of S. Our solution of this problem consists in generalizing the resolution calculus from clauses to cut-free proofs: instead of resolving clauses $C, D \in \mathrm{CL}(\varphi)$, we resolve their projections $\varphi[C]$ and $\varphi[D]$, resulting in a new cut-free proof. We introduce this general resolution principle of proofs in Section 6. The completeness of proof resolution in intuitionistic logic (the derivation yields a cut-free intuitionistic proof) is based on a subsumption principle for proofs which is defined in Section 5. These results yield a method, called CERES-i (defined in Section 8), for cut-elimination in intuitionistic logic: given an intuitionistic proof φ of a sequent S, we first compute the set of projections $\mathcal{P}(\varphi)$ of φ; then we apply proof resolution to $\mathcal{P}(\varphi)$ and derive a cut-free intuitionistic proof ψ of S.

The results are better summarized in the diagram of Fig. 1, where φ represents an **LJ** proof with cuts, φ^t contains only atomic cuts on axioms (obtained from φ via reductive cut-elimination), and $\mathcal{P}(\psi)$ denotes the set of projections of a proof ψ. The rightmost branch represents the CERES-i method proposed, and the leftmost branch is a specific reductive cut-elimination strategy. The middle branch serves as a bridge to show the completeness of CERES-i, i.e., that the final cut-free proof obtained is intuitionistic. This is done via the proof subsumption property, which is indicated in the diagram by the horizontal edges.

In Section 8, we also define a complete refinement of CERES-i which reduces proof search. In Section 10, we compare CERES-i with a refinement of the reductive cut-elimination method shown on the leftmost branch of Fig. 1 and show that CERES-i asymptotically outperforms the reductive method.

In summary, we define a novel method for cut-elimination in intuitionistic logic which unifies methods from proof theory and the resolution calculus. We demonstrate that principles like resolution and subsumption, which are powerful tools in automated deduction, can be generalized to cut-free proofs. We think