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Imaginaries in bounded pseudo real closed fields

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ABSTRACT

The main result of this paper is that if M is a bounded PRC field, then $Th(M)$ eliminates imaginaries in the language of rings expanded by constant symbols. As corollary of the elimination of imaginaries and the fact that the algebraic closure (in the sense of model theory) defines a pregeometry we obtain that the complete theory of a bounded PRC field is superrosy of $U^{þ}$ -rank 1.

1. Introduction

A *pseudo algebraically closed field* (PAC field) is a field M such that every absolutely irreducible affine variety defined over M has an M-rational point. The concept of a PAC field was introduced by J. Ax in [1] and has been extensively studied. The above definition of PAC field has an equivalent model-theoretic version: M is existentially closed (in the language of rings) in each regular field extension of M.

A field M is called *bounded* if for any integer n, M has only finitely many extensions of degree n. Hrushovski showed in [8] that if M is a bounded PAC field, and $\mathcal{L}$ is the language of rings expanded by enough constants, then $Th_{\mathcal{L}}(M)$ eliminate imaginaries.

The notion of PAC field has been generalized by Basarab in [2] and then by Prestel in [13] to ordered fields. Prestel calls a field M a *pseudo real closed field* (PRC field) if M is existentially closed (in the

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language of rings) in each regular field extension to which all orderings of M extend. Remark that if M is a PRC field and has no orderings, then M is a PAC field.

PRC fields were extensively studied in the beginning by L. van den Dries in [14], A. Prestel in [13], M. Jarden in [9–11], S. Basarab in [4] and [3], and others.

In [12], we studied some model theoretic properties of bounded PRC fields. In Theorem 3.17 of [12] we give a good description of definable sets, this description is a generalization to multi-ordered fields of cellular decomposition for real closed fields.

The main result in [12] was that the complete theory of a PRC field M is NTP_2 if and only if M is bounded ([12, Theorem 4.21]). In Theorems 4.22 and 4.30 of [12] we showed that if M is a bounded PRC field with exactly n orders, then the complete theory of M is strong of burden n and resilient. We also gave a description of forking, and showed that the forking depends only on the forking in each real closure ([12, Theorem 4.38]).

The main result of this paper is a generalization to bounded PRC fields of elimination of imaginaries of PAC fields. We show that the complete theory of a bounded PRC field M eliminates imaginaries in the language of rings expanded by constant symbols for an elementary submodel (Theorem 4.11).

As a corollary of the elimination of imaginaries and the fact that the algebraic closure (in the sense of model theory) defines a pregeometry we obtain (Theorem 4.14) that the complete theory of a bounded PRC field is superrosy of $U^{\text{þ}}$ -rank 1. The elimination of imaginaries for the unbounded case is still open, but it is possible that there is not a reasonable language in that case.

The organization of the paper is as follows: In section 2 we give the required preliminaries on pseudo real closed fields. We fix a complete theory T of a bounded PRC field M, where we enrich the language by adding constant symbols for an elementary submodel. As M is bounded, there is $n \in \mathbb{N}$ such that M has exactly n orders. In section 3 we define the theory VO_n in a multi-sorted language $\mathcal{L}_n^*$. To each model of the theory T we associate a model of VO_n as follows: let M be a PRC field endowed with exactly n distinct orderings $\{<_1, \dots, <_n\}$ and let M_i be the real-closure of M with respect to the order $<_i$. Then one associates with $(M, <_1, \dots, <_n)$ the multi-sorted structure $(M, M_1, \dots, M_n)$ where each M_i is endowed with a relation symbol for the order induced by $<_i$ and where one has inclusion maps and their inverse between different sorts. The importance of this theory for the study of PRC fields is that the structure is much more simple, in particular the algebraic and definable closures are almost trivial. This allows us to study the definable sets in an easier way, specifically we can work with the trace of intervals with extremities in the real closures. We show in Theorem 3.3 that the theory VO_n is $\aleph_0$-categorical, has quantifier elimination in $\mathcal{L}_n^*$ and we describe the algebraic and definable closures. We define a notion of multi-interval, this notion coincide with the multi-intervals defined in [12] for PRC fields. We give a notion of canonical decomposition for definable sets in one variable, and we show in Theorem 3.8 that for each definable set in the first sort, there is a unique canonical decomposition. In Theorem 3.11 we prove elimination of imaginaries for the theory VO_n.

Finally in section 4 we work with bounded PRC fields. We generalize Theorem 3.17 of [12] to find a canonical decomposition of definable sets in one variable. An important tool in the proof is the existence of the canonical decomposition for the definable sets in the theory VO_n. In Theorem 4.11 we prove elimination of imaginaries for bounded PRC fields. As a consequence we obtain in Theorem 4.14 that the theory of a bounded PRC field is superrosy.

2. Pseudo real closed fields

In this section we give the required preliminaries on pseudo real closed fields.

Notation and Conventions 2.1. If M is a model of an $\mathcal{L}$-theory T and $A \subseteq M$, then $\mathcal{L}(A)$ denotes the set of $\mathcal{L}$-formulas with parameters in A. If $\bar{a}$ is a tuple of M, we denote by $\text{tp}_{\mathcal{L}}^M(\bar{a}/A)$ ($\text{qftp}_{\mathcal{L}}^M(\bar{a}/A)$) the set of