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Increasing sentences in Simple Type Theory

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We introduce the notion of pseudo-increasing sentence, and prove that all such sentences are decidable by a weak subtheory of Simple Type Theory with infinitely many zero-type elements. We then present the consequences of this result to Quine's theory of "New Foundations" (NF). In particular, we prove the decidability of certain universal-existential sentences, and establish the consistency of a subtheory of NF.

1. Introduction

We remind that a sentence σ is said to be decidable by a theory T if either $T \vdash \sigma$ or $T \vdash \neg\sigma$. The aim of this paper is to establish certain decidability results for weak subtheories of Simple Type Theory (TST) and Quine's theory of "New Foundations" (NF). Our work is centered around the study of the newly introduced set of increasing sentences, namely the set of all sentences

$$\mathrm{Q}_1 x_1^{s_1} \dots \mathrm{Q}_n x_n^{s_n} \phi(x_1^{s_1}, \dots, x_n^{s_n}),$$

where $s_1, \dots, s_n \in \omega$ are the types of the variables $x_1, \dots, x_n$ respectively, $s_1 \leq \dots \leq s_n$, and ϕ is a quantifier-free formula of a specific expanded language of Simple Type Theory (TST). As we will show in our main theorem (Theorem 2.14), all increasing sentences are decidable by a weak subtheory of Simple Type Theory with infinitely many zero-type elements (TST^∞). In fact, by slightly tweaking the definition of

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increasing sentence, we will be able to prove a more general result that applies to a larger class of sentences, which we will call pseudo-increasing. To prove this we will use a modified version of Ehrenfeucht–Fraïssé games.

The above decidability result has a number of consequences to both TST and NF:

(i) All stratified pseudo-increasing sentences (i.e., sentences obtained from pseudo-increasing sentences if we remove the types of all the variables) are decidable by a weak subtheory of NF (Corollary 2.16).

(ii) All stratified existential (and certain stratified universal-existential) sentences of a specific expanded language of NF are decidable by a subtheory of NFO (Corollaries 3.3 and 3.11). This improves a result by Hinnion on the decidability of existential sentences by NFO (see [9]).

(iii) The decidability problem of stratified universal-existential sentences of a specific expanded language of TST can be reduced to that of certain stratified universal-existential sentences (of the same language) with just one universal quantifier (Theorem 3.8). The fact that a slightly weaker version of the reduced problem can be solved (Theorem 3.9) provides further support for Forster's conjecture that all stratified universal-existential sentences are decidable by NFO (see [8]).

(iv) The subtheory of NF that is axiomatized by the universal-existential-increasing theorems of TST (more precisely, by the sentences obtained from these theorems if we remove the types of all the variables) is consistent (Theorem 4.6).

We should note that our work here is related to [13], where we study the decidability of decreasing sentences. A sentence is decreasing if it has the form

$$\mathrm{Q}_1 x_1^{s_1} \dots \mathrm{Q}_n x_n^{s_n} \phi(x_1^{s_1}, \dots, x_n^{s_n}),$$

where $s_1, \dots, s_n \in \omega$ are the types of the variables $x_1, \dots, x_n$ respectively, $s_n \leq \dots \leq s_1$, and ϕ is a quantifier-free formula. So, in a sense, the notion of decreasing sentence is dual to that of an increasing sentence. Apart from [13], we think that the reader interested in this paper should also see the recent work of Dawar, Forster, and McKenzie (see [5]) that establishes very interesting results on the decidability of universal-existential sentences.

Most of the material of this introductory section on TST and NF can also be found in [8] and [13]. Nevertheless, in order to be as precise as possible, we give a brief overview of the basic notions. The main reason for doing this is because there are certain particularities in the definitions that significantly affect the proofs of our results.

The generalization of Ehrenfeucht–Fraïssé games that we will use to prove the decidability of pseudo-increasing sentences is self-contained, but some familiarity with the relevant notions is assumed. If needed, all the necessary background can be found in any basic model theory book that covers back-and-forth games (for instance, see [12]). It is also assumed that the reader is familiar with many-sorted logic and especially with the fact that all results of one-sorted logic can be trivially translated into many-sorted logic (and vice versa). For a detailed way of doing this, we refer the reader to [6].

Lastly, we should mention that our metatheory throughout this text will be ZF.

1.1. *Language and axioms of* TST

The language $\mathcal{L}_{\mathrm{TST}}$ of Simple Type Theory is the many-sorted language of set theory with one binary relation symbol ε and countably many types (or sorts) indexed by ω. Each variable of $\mathcal{L}_{\mathrm{TST}}$ is assigned a unique type, which we indicate by a superscript. The $\mathcal{L}_{\mathrm{TST}}$-formulas are built inductively from the atomic formulas $x^i \varepsilon y^{i+1}$ and $x^i = y^i$ in the usual way. In this text, we mostly work with expansions of $\mathcal{L}_{\mathrm{TST}}$ that include constants or additional relation symbols. For example, in our languages we usually include the

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