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Uniform interpolation and compact congruences

Samuel J. van Gool [a,b,1], George Metcalfe [c,*,2], Constantine Tsinakis [d]

[a] Department of Mathematics, City College of New York, USA
[b] ILLC, University of Amsterdam, The Netherlands
[c] Mathematical Institute, University of Bern, Switzerland
[d] Department of Mathematics, Vanderbilt University, USA

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ABSTRACT

Uniform interpolation properties are defined for equational consequence in a variety of algebras and related to properties of compact congruences on first the free and then the finitely presented algebras of the variety. It is also shown, following related results of Ghilardi and Zawadowski, that a combination of these properties provides a sufficient condition for the first-order theory of the variety to admit a model completion.

1. Introduction

The following remarkable feature of intuitionistic propositional logic IPC was established by A.M. Pitts in [31]. Given any formula $\alpha(\bar{x}, \bar{y})$ of the logic, there exist formulas $\alpha^L(\bar{y})$ and $\alpha^R(\bar{y})$, *left* and *right uniform interpolants* of α with respect to $\bar{x}$, respectively, such that for any formula $\beta(\bar{y}, \bar{z})$,

$$\vdash_{\mathbf{IPC}} \beta \to \alpha \iff \vdash_{\mathbf{IPC}} \beta \to \alpha^L \quad \text{and} \quad \vdash_{\mathbf{IPC}} \alpha \to \beta \iff \vdash_{\mathbf{IPC}} \alpha^R \to \beta.$$

* Corresponding author.
E-mail addresses: samvangool@me.com (S.J. van Gool), george.metcalfe@math.unibe.ch (G. Metcalfe), constantine.tsinakis@vanderbilt.edu (C. Tsinakis).

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Each of the seven intermediate propositional logics admitting the usual Craig interpolation property also admits uniform interpolation; however, there are modal logics, such as S4, that admit Craig interpolation but not uniform interpolation (see [15] for details and references).

The main aim of this paper is to study uniform interpolation in the setting of universal algebra. Instead of a propositional logic, we consider a variety of algebras: a class of algebraic structures of the same signature that is defined by equations (equivalently, closed under homomorphic images, subalgebras, and direct products). Although there may exist faithful translations between the variety and a propositional logic (e.g., Boolean algebras and classical logic, Heyting algebras and intuitionistic logic, MV-algebras and Łukasiewicz logic), this need not be the case. In particular, the signature may not contain a 'suitable' implication connective. We therefore focus here on consequences in the variety between a set of equations on the left and a single equation on the right. The most natural interpolation property in this setting is *deductive interpolation* (studied in [11,12,18,20,23,28,29,34]), which coincides with Craig interpolation only in the presence of a suitable deduction theorem. Indeed, a variety with the congruence extension property admits deductive interpolation if, and only if, it has the amalgamation property (see [23] for proofs and references). In this paper, we define left and right uniform interpolation properties for a variety and obtain corresponding algebraic characterizations. We also obtain algebraic characterizations for uniform versions of the Maehara interpolation property studied in [11,23,34].

The starting point for our study of uniform interpolation is the extensive work on interpolation and model completions by Ghilardi and Zawadowski, collected in the monograph [15]. These authors studied category-theoretic properties of varieties that correspond to propositional logics with (left and right) uniform interpolation, establishing these properties for certain varieties of Heyting and modal algebras. Our motivations here for supplementing this category-theoretic view of uniform interpolation with a universal algebraic perspective are two-fold. First, we define left and right uniform interpolation as specific properties of a variety, whereas in [15] these arise as combinations of other properties. This allows us to identify new examples of varieties with and without uniform interpolation, not restricted to intermediate and modal logics, including groups, MV-algebras, implicative semilattices, Sugihara monoids, and bounded lattices. Second, our algebraic perspective exhibits connections between uniform interpolation and known properties in equational logic and universal algebra. In particular, we obtain uniform versions of Maehara interpolation and a better understanding of the connection between uniform interpolation and amalgamation.

A further logical motivation for studying uniform interpolation lies in its relationship to the notion of a model completion, which originated in the groundbreaking work on model-theoretic algebra of A. Robinson [32]. A model completion of a first-order theory axiomatizes the class of algebras in which 'all potentially solvable equations possess solutions' and always has quantifier elimination. The prototypical example is the theory of fields, whose model completion is the theory of algebraically closed fields. Ghilardi and Zawadowski showed in [15] that category-theoretic properties of varieties of algebras for intermediate and modal logics with uniform interpolation imply the existence of a model completion for the first-order theory of the variety. Building on their work, we relate the algebraic uniform interpolation properties introduced in this paper to the existence of a model completion. This approach is also related to the use in [21,22] of model-theoretic methods (in particular, quantifier elimination) to establish the amalgamation property for certain varieties of semilinear commutative residuated lattices.

The key notions needed for this algebraic view of uniform interpolation turn out to be compact (i.e., finitely generated) congruences and pairs of adjoint maps between them. The importance of compact congruences was already implicit in [15], where they appear as 'regular monomorphisms in the opposite of a category of finitely presented algebras'. A notable conceptual contribution of our work is that we view these notions as central to the algebraic study of uniform interpolation. Compact congruences are needed in order to obtain a stable theory of uniform interpolation when venturing outside the realm of Heyting and modal algebras considered in [15], where compact congruences may be harmlessly identified with certain elements

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