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Ordinals and graph decompositions

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ABSTRACT

The theory of simplicial graph decompositions studies the infinite graphs that are built from a sequence of irreducible graphs which are attached together at complete subgraphs. In this paper, we study the logical complexity of deciding if a graph is prime decomposable. A large part of this analysis involves determining which ordinals must appear in these types of decompositions.

A result of Diestel says that every countable simplicial tree decomposition can be rearranged to have length at most ω. We show that no such ordinal bound can be found for the lengths of non-tree decompositions. More generally, we show that for each ordinal σ, there is a decomposable graph whose shortest simplicial decomposition has length exactly σ. Adapting this argument, we show that the index set of decomposable computable graphs DECOMP is Π^1_1 hard by showing that WO is 1-reducible to DECOMP.

1. Introduction

The theory of graph decompositions studies a certain structure that occurs throughout graph theory. In many arguments, such as some proofs for Kuratowski's Theorem and in some cases of Hadwiger's conjecture (see Chapters 4.4 and 7.3 of [3]), a theorem is proved by showing that a class of graphs is built from a certain set of "nice" subgraphs, which are pasted together at complete subgraphs. Because "pasting at complete subgraphs" preserves many graph theoretic-properties, it is possible to lift facts about the starting set of graphs to the full class of graphs. A sample application is given in Section 1.1.

In the theory of graph decompositions [2], our interest is in the existence, uniqueness, and structure of the decompositions themselves.

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In the *Note to the reader* of [2], Diestel says

"The axiom of choice will be assumed throughout the book, and we shall generally steer well clear of the pitfalls of set theory."

It is natural to ask *exactly* how much logical strength is being used. In this paper, we will show that all the ordinals appear in the study of graph decompositions. Adapting this argument, we show that the index set of decomposable computable graphs DECOMP is Π^1_1 hard by showing WO is 1-reducible to DECOMP.

In another paper [4] we will examine the reverse mathematics of particular theorems about graph decompositions. In particular, we will show that Halin's theorem, one of the theory's main existence theorems, is equivalent to ACA_0. [4] will also discuss additional upper bounds for defining tree-decomposability.

Note that the results in the current paper imply that any reverse mathematical analysis of general (non-tree) decompositions will need to pay careful attention to ordinal length. This is in contrast to other work on reverse mathematics of graph theory, even concerning theorems which are equivalent to ATR_0 or Π^1_1-CA, such as the versions of König's duality theorem studied by [1,7].

1.1. A motivating graph-theoretic example

Before formally defining a graph decomposition, it may be helpful to illustrate a practical application of decomposing certain graphs.

This illustration requires two standard definitions, neither of which will be used in the remainder of the paper. First, an r coloring of a graph G is an assignment of r colors to the vertices of G so that no adjacent vertices are given the same color. It is clear that K^r, the complete graph on r vertices, cannot have an $r-1$ coloring. Second, a graph $H \subseteq G$ is called a minor of G if H can be formed from G by deleting edges and vertices and by contracting edges.

Hadwiger's Conjecture states that for each $r > 0$, if G doesn't have a K^r minor, then G has a $r-1$ coloring. This conjecture has been proved for $r \leq 6$.

Proofs of several cases can be given using the ideas behind graph decompositions. The key lemma for the $n = 4$ case is the following.

Proposition 1 *(Part of Proposition 7.3.1. of [3]). Every edge-maximal G without a K^4 minor can be constructed recursively by pasting triangles at K^2s.*

That is, the graphs of interest can be built as the union of triangles B_λ, ordered so that each point of attachment $\bigcup_{\rho<\lambda} B_\rho \cap B_\lambda$ is the subgraph K^2.

From this proposition, it is not difficult to prove Hadwiger's Conjecture for $r = 4$. First, the factors are triangles which are individually 3-colorable. Second, pasting two graphs together at a K^2 preserves 3-colorability. Finally, every graph without a K^4 minor lives inside an edge maximal graph without a K^4 minor, and removing edges preserves 3-colorability.

The $n = 5$ case has a similar proof. The key component is the following.

Theorem 2 *(Wagner. Theorem 7.3.4 of [3]). Every edge maximal G without a K^5 minor can be constructed recursively by pasting plane triangulations and copies of a certain 4-colorable graph W together at K^2s and K^3s.*

Applying the 4 color theorem, and arguing as above, proves Hadwiger's conjecture for $r = 5$. See Chapter 7.3 of [2] for full details. For the remainder of the paper, we turn our attention to the *theory* of graph decompositions.