ARTICLE IN PRESS
APAL:2546

Annals of Pure and Applied Logic ••• (••••) •••–•••

Contents lists available at ScienceDirect

Annals of Pure and Applied Logic

www.elsevier.com/locate/apal

Computable neighbourhoods of points in semicomputable manifolds

Zvonko Iljazović *, Lucija Validžić

Department of Mathematics, Faculty of Science, University of Zagreb, Croatia

ARTICLE INFO

Article history:
Received 7 April 2016
Received in revised form 9 July 2016
Accepted 11 October 2016
Available online xxxx

MSC:
03D78
03F60
68Q05
57N99

Keywords:
Computable metric space
Computable point
Computable neighbourhood
Manifold
Polyhedron

ABSTRACT

We examine conditions under which a semicomputable set in a computable metric space contains computable points. We prove that computable points in a semicomputable set S are dense if S is a manifold (possibly with boundary) or S has the topological type of a polyhedron. Moreover, we find conditions under which a point in some set has a computable compact neighbourhood in that set. In particular, we show that a point x in a semicomputable set has a computable compact neighbourhood if x has a neighbourhood homeomorphic to Euclidean space.

1. Introduction

A real number $x \in \mathbb{R}$ is called computable if for an arbitrary $k \in \mathbb{N}$ we can effectively find a rational number which approximates x up to 2^{-k}. We say that a set $S \subseteq \mathbb{R}$ is co-computably enumerable (co-c.e.) if we can effectively cover the complement $\mathbb{R} \setminus S$ by rational open intervals, i.e. if $\mathbb{R} \setminus S = \bigcup_{i \in \mathbb{N}} \langle a_i, b_i \rangle$, where (a_i) and (b_i) are computable sequences of rational numbers. Equivalently, S is co-c.e. if $\mathbb{R} \setminus S$ can be effectively covered by open balls with rational centers and radii (with respect to the Euclidean metric on $\mathbb{R}$).

The question is: does each nonempty co-c.e. set contain a computable point? The answer is negative: there exists a nonempty co-c.e. set without a computable point (see [21]). However, if S is a co-c.e. set which has an isolated point, then that point must be computable. In particular, each point in a finite co-c.e. set is computable.

* Corresponding author.
E-mail address: zilj@math.hr (Z. Iljazović).

http://dx.doi.org/10.1016/j.apal.2016.10.015

The notions of a computable point and a co-c.e. set can be easily generalized to n-dimensional Euclidean space $\mathbb{R}^n$ or to any computable metric space (X, d, α). Although a co-c.e. set need not contain a computable point, the question is what are conditions under which a co-c.e. set contains a computable point. A motivation for this lies in the fact that $S \subseteq \mathbb{R}^n$ is co-c.e. if and only if $S = f^{-1}(\{0\})$ for some computable function $f : \mathbb{R}^n \to \mathbb{R}$, hence an answer to the previous question would provide conditions under which a computable function $\mathbb{R}^n \to \mathbb{R}$ has a computable zero-point.

As noted, if a one-point set $\{x\}$ is co-c.e. in $\mathbb{R}$, then x is a computable point. The same holds in any Euclidean space $\mathbb{R}^n$. However, this need not hold in a general computable metric space. This is the reason why we will, instead of co-c.e. sets, observe a stronger notion, the notion of a semicomputable set. A compact set S is semicomputable if we can effectively enumerate all rational open sets (i.e. sets which are finite unions of rational open balls) which cover S. More generally, a closed set S is semicomputable if $S \cap B$ is a compact set for each rational closed ball B and if we can effectively enumerate all rational open sets which cover $S \cap B$ uniformly in B. In Euclidean space a set is co-c.e. if and only if it is semicomputable. In a general computable metric space a semicomputable set is co-c.e., but the converse does not hold always. However in computable metric spaces which have compact closed balls and the effective covering property these two notions coincide.

In contrast to the fact that there exists (in some computable metric space) a co-c.e. one-point set $\{x\}$ such that x is incomputable, in any computable metric space the semi-computability of $\{x\}$ implies that x is computable. Therefore it makes more sense to investigate conditions under which in a general computable metric space a semicomputable set contains a computable point rather than to observe the same problem for co-c.e. sets.

What kind of conditions can we expect here? It is known that topology plays an important role in this view. Actually, there are some topological conditions under which a semicomputable set is computable. That a nonempty compact set is computable means that it can be effectively approximated by a finite set of points with rational coordinates with arbitrary precision. Each computable set contains computable points, moreover they are dense in it. Regarding conditions under which the implication

$$S \text{ semicomputable} \Longrightarrow S \text{ computable} \tag{1}$$

holds, the pioneer work in this area was made by Miller [16] who showed that (1) holds in Euclidean space if S is a topological sphere or if S is a cell with computable boundary sphere. These results were later generalized in [10] by showing that (1) holds in any computable metric space if S is a compact manifold with computable boundary (in particular if S is a compact boundaryless manifold). In general, however, (1) does not hold if S is a compact manifold (with incomputable boundary): there exists a semicomputable line segment in $\mathbb{R}$ which is not computable [16]. Therefore it makes sense to ask the question: does every semicomputable (compact) manifold with boundary S contain a computable point? In [16] it was proved that every semicomputable cell S in Euclidean space contains computable points, moreover they are dense in S.

In this paper we show that this holds for any semicomputable manifold, i.e. if S is a semicomputable manifold (possibly with boundary) in a computable metric space, then the computable points which lie in S are dense in S.

The fact that computable points are dense in some set S is related to the fact that S is computably enumerable (c.e.). A closed set S in a computable metric space is computably enumerable if we can effectively enumerate all rational balls in this space which intersect S. If S is complete (as a subspace of the ambient metric space), then S is c.e. if and only if there exists a computable sequence which is dense in S. This gives rise to the following definition. We will say that a closed set S is weakly c.e. if the computable points of S are dense in S. So one of the main problems that we study in this paper can be formulated in this way: under what conditions the implication

Please cite this article in press as: Z. Iljazović, L. Validžić, Computable neighbourhoods of points in semicomputable manifolds, Ann. Pure Appl. Logic (2016), http://dx.doi.org/10.1016/j.apal.2016.10.015